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COMPLEX ANALYSIS

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SYLLABUS

UNIT I

Functions of a Complex variable –Limits –Theorem on limits –Continuity Derivatives –Differentiation formulas –Cauchy Riemann equation –conditions for differentiability –Polar coordinates–Analytic functions–Harmonic functions

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UNIT III

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Recommended Text

S. Arumugam, A. Thangapandi Isaac and A.Somasundaram, Complex Analysis, SciTech, 2014

JMMA62: COMPLEX ANALYSIS

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UNIT I

Functions of a Complex variable –Limits –Theorem on limits –Continuity Derivatives –Differentiation formulas –Cauchy Riemann equation –conditions for differentiability –Polar coordinates–Analytic functions–Harmonic functions

Chapter 1: Sections -1.1 to 1.8

1. Analytic Functions

Introduction

In this chapter we study in detail the concepts of limit and continuity for functions a complex variable. We also introduce the notion of differentiability for functions a complex variable and see how the derivative of a complex function of one complex, variable sometimes behaves like the derivative of a real function of one real variable and other times is comparable to the partial derivatives of a real function of two variables.

1.1. Functions of a Complex Variable:

We use the letters z and w to denote complex variables. Thus, to denote a complex value function of a complex variable we use the notation $w = f(z)$. Throughout this chapte we shall consider functions whose domain of definition is a region of the complex plane

The function $w = iz + 3$ is defined in the entire complex plane. The function $w = \frac{1}{z^2+1}$ is defined at all points of the complex plane except a $z = \pm i$.

The function $w = |z|$ is defined in the entire complex plane and this is a real value function of the complex variable z.

If $a_0, a_1, \cdots, a_n$ are complex constants the function $P(z) = a_0 + a_1 z + \cdots + a_n z^r$ is defined in the entire complex plane and is called a polynomial in z.

If $P(z)$ and $Q(z)$ are polynomials the quotient $\frac{P(z)}{Q(z)}$ is called a rational function anc it is defined for all z with $Q(z) \neq 0$.

The function $f(z) = x^4 + y^4 + i(x^2 + y^2)$ is defined over the entire complex plane. In general, if $u(x, y)$ and $v(x, y)$ are real valued functions of two variables both definec on a region S of the complex plane then

$f(z) = u(x, y) + iv(x, y)$ is a complex valued function defined on S.

Conversely each complex function $w = f(z)$ can be put in the form

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are real valued functions of the real variables x and y.

$u(x, y)$ is called the real part and $v(x, y)$ is called the imaginary part of the function $f(z)$.

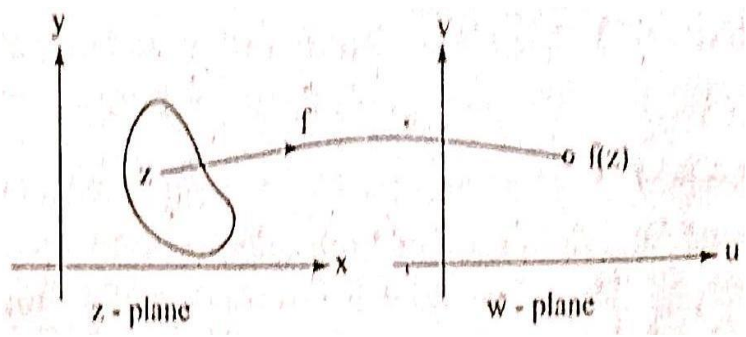
For example, $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Thus, a complex function $m = f(z)$ can be viewed as a function of the complex variable z or as a function of two real variables x and y.

To have a geometric representation of the function $w = f(z)$ it is convenient to draw separate complex planes for the variables z and w so that corresponding to each point $z = x + iy$ of the $z - p$ plane there is a point

$w = u + iv$ in the w-plane.



Exercises.

1. Express each of the following functions in the form $u(x, y) + iv(x, y)$

(i) $u = z^3$

(ii) $w = 2\bar{z}^2 + 1$

(iii) $w = \frac{1}{z}$

(iv) $w = \frac{z}{1+z}$

(v) $w = z + \frac{1}{z}$

(vi) $w = z\bar{z}$

Answers

(i) $(x^3 - 3xy^2) + i(3x^2y - y^3)$

(ii) $(2x^2 - 2y^2 + 1) - i(4xy)$

(iii) $\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

(iv) $\frac{x^2+x+y^2}{(x+1)^2+y^2} + \frac{iy}{(x+1)^2+y^2}$

(v) $\frac{x(x^2+y^2+1)}{x^2+y^2} + i\,\frac{y(x^2+y^2-1)}{x^2+y^2}$

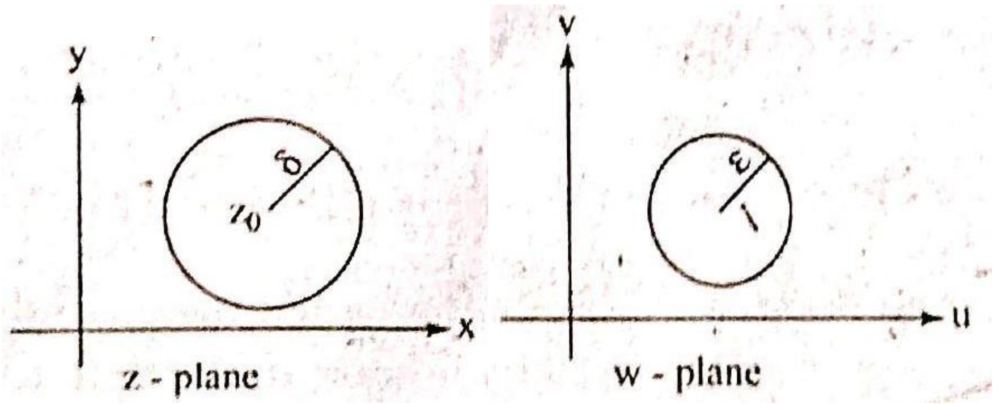
(vi) $x^2 + y^2$

1.2 Limits

Let $w = f(z)$ be a function defined in some region containing a point z_0 except perhaps at the point z_0. It may happen that as z approaches z_0 the value $f(z)$ of the function is arbitrarily close to a complex number l. Then we say that the limit of the function $f(z)$ as z approaches z_0 is l. This idea is expressed in a precise form in the following definition.

Definition:

A function $w = f(z)$ is said to have the limit l as z tends to z_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$ In this case we write $\lim_{z \to z_0} f(z) = l$.



Geometrically the definition states that given any open disc with center l and radius, there exists an open dise with centre z_0 and radius δ such that for every point $z(+z_0)$ in the disc $|z - z_0| < \delta$ the image $w = f(z)$ lies in the disc $|w - l| < \varepsilon$.

Remark 1.

The above definition does not give any method of determining the limit / and it only provides a means of testing whether l is the limit of $f(z)$ as $z \to z_0$.

Remark 2.

The condition $0 < |z - z_0| < \delta$ excludes the point $z = z_0$ from consideration Hence the definition of limit employs only values of z in some disc

$|z - z_0| < \delta$ other than z_0.

Therefore, the value of $f(z)$ at z_0 is immaterial and in fact to consider the limit of $f(z)$ as $z \to z_0, f(z)$ need not even be defined at z_0. Even if $f(z_0)$ is defined it is not necessary that $\lim_{z \to z_0} f(z) = f(z_0)$.

Lemma 1:

When the limit of a function $f(z)$ exists as z tends to z_0 then the limit has a unique value.

Proof:

Suppose that $\lim_{z \to z_0} f(z)$ has two values l_1 and l_2.

Then given $\varepsilon > 0$ there exists δ_1 and $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \varepsilon/2 \text{ and}$$
$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \varepsilon/2.$$

Now let $\delta = \min\{\delta_1, \delta_2\}$.

Then if $0 < |z - z_0| < \delta$ we have

$$\begin{aligned} |l_1 - l_2| &= |l_1 - f(z) + f(z) - l_2| \\ &\leq |f(z) - l_1| + |f(z) - l_2| \quad \text{(using triangle inequality)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $l_1 - l_2 \mid = 0$ so that $l_1 = l_2$.

Example 1:

Let $f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & \text{if } z = i \end{cases}$

As z approaches $i, f(z)$ approaches $i^2 = -1$.

Hence, we expect that $\lim_{z \to i} f(z) = -1$.

To prove that we must show that given $\varepsilon > 0$ there exists $\delta >$∩ such that $0 <$ $|z - i| < \delta \Rightarrow |z^2 + 1| < \varepsilon$.

Now, $|z^2 + 1| = |(z + i)(z - i)|$

$$= |z + i||z - i| \dots \dots \dots . . (1)$$

Note that if we can find a $\delta > 0$ satsifying the requirements of the definition then we can choose another $\delta \leq 1$ satisfying the requirements of the definition.

$$\begin{aligned} \text{Now } 0 < |z - i| < 1 \Rightarrow |z + 1| &= |z - i + 2i| \\ &\leq |z - 1| + |2i| \\ &< 1 + 2 = 3 \end{aligned}$$

$$|z + i| < 3.$$

Using this in (1) we obtain $0 < |z - i| < 1$.

$$\Rightarrow |z^2 + 1| < 3|z - i|$$

Hence if we choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ we get

$0 < |z - i| < \delta \Rightarrow |z^2 + 1| < \varepsilon.$

$\therefore \lim_{z \to i} f(z) = -1.$

Example 2:

$\lim_{z \to 2} \frac{z^2-4}{z-2} = 4.$

Let $f(z) = \frac{z^2-4}{z-2}$. Hence $f(z)$ is not defined at $z = 2$ and when $z \neq 2$ we have

$f(z) = \frac{(z+2)(z-2)}{z-2} = z + 2.$

$\therefore |f(z) - 4| = |z + 2 - 4| = |z - 2|$ when $z \neq 2$.

Now given $\varepsilon > 0$, we choose $\delta = \varepsilon$.

Then $0 < |z - 2| < \delta \Rightarrow |f(z) - 4| < \varepsilon$.

$\therefore \lim_{z \to 2} f(z) = 4.$

Remark.

For a function of one real variable there are only two directions along which a point x can travel and tend to a point x_0. Hence, we distinguish between left limit and right limit at a point x_0 and the limit at x_0 exists if and only if the left and right limit at x_0 exist and are equal.

For functions of a complex variable infinitely many modes of approaches are possible for a point z to tend to a point z_0. If a function $f(z)$ approaches to two different values as z tends to z_0 along two different paths, then $\lim_{z \to z_0} f(z)$ does not exist.

Example 3:

The function $f(z) = \frac{\bar{z}}{z}$ does not have a limit as $z \to 0$.

$$f(z) = \frac{\bar{z}}{z} = \frac{x - iy}{x + iy}$$

Suppose $z \to 0$ along the path $y = mx$.

Along this path $f(z) = \frac{x-imx}{x+imx} = \frac{1-im}{1+im}$ as $x \neq 0$.

Hence if $z \to 0$ along the path $y = mx, f(z)$ tends to $\frac{1-im}{1+im}$ which is different f_{0y} different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Example 4:

Let $f(z) = \frac{x^2y^2}{(x+y^2)^3}, z \neq 0$. Then $f(z)$ does not have a limit as $z \to 0$.

Along the parabola $y^2 = mx$ we have $f(z) = \frac{mx^3}{(x+mx)^3} = \frac{m}{(1+m)^3}$.

Hence if $z \to 0$ along the parabola $y^2 = mx, f(z)$ tends to $\frac{m}{(1+m)^3}$ which depends on m.

Hence $f(z)$ does not have a limit as $z \to 0$.

Exercises.

1. Use the definition of limit to prove the following

   (a) $\lim_{z \to 0} c = c$ where c is any constant.

   (b) $\lim_{z \to z_0} (az + b) = az_0 + b$

   (c) $\lim_{z \to z_0} z^2 = z_0^2$

   (d) $\lim_{z \to z_0} \bar{z} = \bar{z}_0$

   (e) $\lim_{z \to z_0} \text{Re}z = \text{Re}z_0$.

2. Prove that each of the following functions $f(z)$ does not have a limit as $z \to 0$.

   (a) $f(z) = \frac{\sqrt{|xy|}}{x+iy}, z = x + iy \neq 0$.

   (b) $f(z) = \frac{-ix^3y}{x^6+y^2}, z \neq 0$

   (c) $f(z) = \frac{xy}{x^2+y^2}, z \neq 0$.

we now formulate the definition of $\lim_{z \to z_0} f(z) = l$ where z_0 or l is infinite.

Definition:

We say $\lim_{z\to\infty} f(z) = l$ if given $\varepsilon > 0$ there exists a number $m > 0$ such that $|z| > m \Rightarrow |f(z) - l| < \varepsilon$.

We say that $\lim_{z\to z_0} f(z) = \infty$ if for given $n > 0$ there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z)| > n$.

We say that $\lim_{z\to\infty} f(z) = \infty$ if for given $n > 0$ there exists $m > 0$ such that $|z| > m \Rightarrow |f(z)| > n$.

1.3 Theorems on Limit:

We state without proof the following theorem on the limits of sum, product and quotient of two functions. The proof is analogous to that of real functions.

Theorem 1:

Let f and g be two functions whose limits at z_0 exist.

Let $\lim_{z\to z_0} f(z) = l$ and $\lim_{z\to z_0} g(z) = m$.

Then (i) $\lim_{z\to z_0} [f(z) + g(z)] = l + m$

(ii) $\lim_{z\to z_0} f(z)g(z) = lm$

(iii) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{1}{m}$ provided $m \neq 0$.

Theorem 2:

(i) If $\lim_{z\to z_0} f(z) = l$ then $\lim_{z\to z_0} \overline{f(z)} = \bar{l}$.

(ii) If $\lim_{z\to z_0} f(z) = l$, then $\lim_{z\to z_0} |f(z)| = |l|$.

(iii) $\lim_{z\to z_0} f(z) = l$ iff $\lim_{z\to z_0} \mathrm{Re} f(z) = \mathrm{Re} l$ and $\lim_{z\to z_0} \mathrm{Im} f(\bar{z}) = \mathrm{Im}(l)$.

Proof:

(i) Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$.

Now $|\overline{f(z)} - \bar{l}| = |\overline{f(z) - l}| = |f(z) - l|$

Hence $0 < |z - z_0| < \delta \Rightarrow |\overline{f(z)} - \bar{l}| < \varepsilon$ so that $\lim_{z \to z_0} \overline{f(z)} = \bar{l}$.

(ii) $||f(z)| - |l|| \leq |f(z), -l|$ and hence

$0 < |z - z_0| < \delta \Rightarrow \|f(z)| - |l\| < \varepsilon$.

$$\therefore \lim_{z \to z_0} |f(z)| = |l|.$$

(iii) Let $\lim_{z \to z_0} f(z) \mid= l$.

Since $\text{Re}f(z) = \frac{1}{2}[f(z) + f(\bar{z})]$ we have

$$\lim_{z \to -0} \text{Re}f(z) = \frac{1}{2}\left[\lim_{z \to z_0} f(z) + \lim_{z \to z_0} f\overline{(z)}\right]$$
$$= \frac{1}{2}(l + \bar{l})$$
$$= \text{Re}l$$

Similarly, $\lim_{z \to -0} \text{Im}f(z) = \text{Im}l$.

Conversely, let $\lim_{z \to z_0} \text{Re}f(z) = \text{Re}l$ and let $\lim_{z \to z_0} \text{Im}f(z) = \text{Im}l$. Since $f(z) =$ $\text{Re}f(z) + i\text{Im}f(z)$ it follows that $\lim_{z \to z0} f(z) = \text{Re}l + i\text{Im}l = l$.

Remark. It follows immediately from the definition of limit that $\lim_{z \to z_0} z = z_0$ and $\lim_{z \to z_0} c = c$ where c is any constant.

Hence using (i) and (ii) of theorem 2.1 we see that for any polynomial $P(z) =$ $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $\lim_{z \to z_0} P(z) = P(z_0)$.

Exercises.

1. Evaluate the following limits using theorems 1 and 2

   (i) $\lim_{z \to 2i} (2x + iy^2)^2$

   (ii) $\lim_{z \to 1-i} [x + i(2x + y)]$

   (iii) $\lim_{z \to 1+i} (z^2 - 5z + 10)$

   (iv) $\lim_{z \to -2i} \frac{(z+3)(z-4)}{z^2+5z+9}$

(v) $\lim_{z \to -i} \frac{\bar{z}+z^2}{1-\bar{z}}$

(vi) $\lim_{z \to z_0} \frac{1}{z^n}, z_0 \neq 0.$

Answers.

(i) -16 (ii) $1 + i$ (iii) $5 - 3i$

(iv) $-\frac{2(3-2i)(4+2i)}{5(1-2i)}$ (v) -1 (vi) $\frac{1}{z_0^n}$.

1.4. Continuous Functions

Definition:

Let f be a complex valued function defined on a region D of the complex plane. Let $z_0 \in D$. Then f is said to be continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Thus f is continuous at z_0 if given $\varepsilon > 0$ there exisis a $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

f is said to be continuous in D if it is continuous at each point of D.

The following are immediate consequences of the corresponding results on limits given in sec 1.3 theorems 1 and 2.

Theorem 1:

(i) If f and g are continuous at z_0 then $f + g, fg$ and $\bar{f}$ are continuous at z_0 and f/g is continuous at z_0 if $g(z_0) \neq 0$.

(ii) If f is continuous at z_0 then $|f|$ is also continuous at z_0.

(iii) If f is continuous at z_0 iff Ref and Imf are continuous at z_0.

(iv) Any polynomial $P(z)$ is continuous at each point of the complex plane and any rational functin $\frac{P(z)}{Q(z)}$ is continuous at all points where $Q(z) \neq 0$.

1.5. Differentiability:

Definition:

Let f be a complex function defined in a region D and let $z \in D$. Then f is said to be differentiable at z if $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists and is finite. This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ and is called the derivative of $f(z)$ at z.

The function is said to be differentiable in D if it is differentiable at all points of D.

Example 1.

The function $f(z) = z^2$ is differentiable at every point and $f'(z) = 2z$.

Proof:

$$\frac{f(z+h)-f(z)}{h} = \frac{(z+h)^2 - z^2}{h}$$
$$= 2z + h$$

Hence $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h \to 0} (2z + h)$

$$= 2z$$

$\therefore f'(z) = 2z.$

Example 2:

The function $f(z) = \bar{z}$ is nowhere differentiable.

Proof:

$$\frac{f(z+h)-f(z)}{h} = \frac{\overline{(z+h)} - \bar{z}}{h}$$

$$= \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \frac{\bar{h}}{h}$$

$\lim_{h \to 0} \frac{\bar{h}}{h}$ does not exist (refer example 3 of 1.2).

$\therefore f(z) = \bar{z}$ is nowhere differentiable.

Remark 1.

If $f(z)$ is differentiative at a point z then it is continuous at that point.

Proof:

$$\lim_{h\to 0}[f(z+h)-f(z)] = \lim_{h\to 0}\left[\frac{f(z+h)-f(h)}{h}\right] \times \lim_{h\to 0} h$$
$$= f'(z) \times 0$$
$$= 0$$

$\therefore \lim_{h\to 0} f(z+h) = f(z)$ so that f is continuous at z.

The converse of the above result is not true.

For example, $f(z) = \bar{z}$ is continuous everywhere but it is nowhere differentiable (refer example 2).

The definition of derivative for complex functions is identical to the definition for real functions and the following formal rules of differentiation are true for complex functions also and the proof is left as an exercise.

Theorem 1:

Let $f(z)$ and $g(z)$ be differentiable at a point z. Then

(i) $(f+g)'(z) = f'(z) + g'(z)$.

(ii) $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$.

(iii) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{[g(z)]^2}$ provided $g(z) \neq 0$.

(iv) Suppose g is differentiable at z and f is differentiable at $g(z)$.

Let $F(z) = f(g(z))$. Then $F'(z) = f'(g(z))g'(z)$. (This is the usual chain rule for the derivative of composite functions).

(v) Let n be any positive integer. The function $f(z) = z^n$ is differentiable at every point and $f'(z) = nz^{n-1}$.

(vi) The polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is differentiable at every point and $P'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$.

(vii) If n is a negative integer $f(z) = z^n$ is differentiable at every point $z \neq 0$ and $f'(z) = nz^{n-1}$.

Exercises.

1. Find the derivative of the following functions.

   (i) $z^2 + 3z + 1$

(ii) $\frac{z+1}{2z+3}\left(z \neq -\frac{3}{2}\right)$

(iii) $\frac{(2+z^3)^4}{z-1}; z \neq 1.$

2. Prove that $f(z) = \frac{z-1}{z+1}$ is differentiable at every point $z \neq -1$ and find $f'(z)$.

3. Prove that $f(z) = \text{Re}z$ is not differentiable at any point.

4. Prove that $f(z) = \text{Im}z$ is not differentiable at any point.

Answers.

(ii) $\frac{1}{(2z+3)^2}$

(iii) $\frac{(2+z^3)^3(11z^3-12z^2-2)}{(z-1)^2}$

1. (i) $2z + 3$ 2. $\frac{2}{(z+1)^2}$.

1.6. The Cauchy-Riemann Equations:

The existence of the derivative of a complex function of a complex variable $f(z)$ requires $\frac{f(z+h)-f(z)}{h}$ to approach to the same limit as $h \to 0$ along any path. This has some far-reaching consequences. In this section we derive some important properties of the real and imaginary parts of the differentiable function $f(z) = u(x,y) + iv(x,y)$.

Theorem 1:

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives

$u_x(x_0,y_0), u_y(x_0,y_0), v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ at (x_0,y_0) and these partial derivatives satisfy the Cauchy-Riemann equations (C.R equations) given by

$u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$.

Also, $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$

$= v_y(x_0,y_0) - iu_y(x_0,y_0)$.

Proof:

Since $f(z) = u(x,y) + iv(x,y)$ is differentiable at

$z_0 = x_0 + iy_0 \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and hence the limit is independent of the path in which h approahces zero.

Let $h = h_1 + ih_2$.

Now $\frac{f(z_0+h)-f(z_0)}{h}$

$$= \frac{u(x_0 + h_1, y_0 + h_2) + iv(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) - iv(x_0, y_0)}{h_1 + ih_2}$$

$$= \left[\frac{u(x_0) + h_1, y_0 + h_2) - u(x_0, y_0)}{h_i + ih_2}\right] + i\left[\frac{v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0)}{h_1 + ih_2}\right]$$

Suppose $h \to 0$ along the real axis so that $h = h_1$.

$$\text{Then } f'(z_0) = \lim_{h_1 \to 0}\left[\frac{f(z_0 + h_1) - f(z_0)}{h_1}\right]$$

$$= \lim_{h_1 \to 0}\left[\frac{u(x_0 + h_1, y_0) - u(x_0, y_0)}{h_1}\right] + i\lim_{h_1 \to 0}\left[\frac{v(x_0 + h_1, y_0) - v(x_0, y_0)}{h_1}\right]$$

$$= u_x(x_0, y_0) + iv_x(x_0, y_0) \qquad \ldots\ldots\ldots\ldots(1)$$

Now, suppose $h \to 0$ along the imaginary axis so that $h = ih_2$.

$$\therefore f'(z_0) = \lim_{ih_2 \to 0}\left[\frac{f(z_0 + h_2) - f(z_0)}{ih_2}\right]$$

$$= \lim_{h_2 \to 0}\left[\frac{u(x_0, y_0 + h_2) - u(x_0, y_0)}{ih_2}\right]$$

$$+ i\lim_{h_2 \to 0}\left[\frac{v(x_0, y_0 + h_2) - v(x_0, y_0)}{ih_2}\right]$$

$$= \left[\frac{u_y(x_0, y_0)}{i}\right] + i\left[\frac{v_y(x_0, y_0)}{i}\right]$$

$$= \frac{1}{i}u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= -iu_y(x_0, y_0) + v_y(x_0, y_0)$$

From (1) and (2) we get

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$$

Equating real and imaginary parts we get

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

Remark 1:

Since $f'(z) = u_x + iv_x = v_y - iu_y$ we have

$|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2$

Also $|f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$

Further $|f'(z)|^2 = u_x v_y - u_y v_x$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$
$$= \frac{\partial(u, v)}{\partial(x, y)}$$

Remark 2:

The Cauchy-Riemann equations provide a necessary condition for differentiability at a point. Hence if the C.R equations are not satisfied for a complex function at any point then we can conclude that the function is not differentiable.

For example, consider the function $f(z) = \bar{z} = x - iy$.

Here $u(x, y) = x$ and $v(x, y) = -y$.

$\therefore u_x(x, y) = 1$ and $v_y(x, y) = -1$.

$\therefore u_x \neq v_y$ so that C.R equations are not satisfied at any point z.

Hence the function $f(z) = \bar{z}$ is nowhere differentiable.

Remark 3.

The C.R equations are not sufficient for differentiability at a point as shown in the following examples.

Example 1:

Let $f(z) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Here $u(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

and $v(x,y) = 0$.

Now, $$u_x(0,0) = \lim_{h\to 0}\left[\frac{u(h,0)-u(0,0)}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{0-0}{h}\right] = 0$$

Similarly, $u_y(0,0) = 0$.

Also $v_x(0,0) = 0$ and $v_y(0,0) = 0$.

Hence the C.R equations are satisfied at $z = 0$.

Now, along the path $y = mx$

$$f(z) = \frac{xmx}{x^2+m^2x^2} = \frac{m}{1+m^2} \text{ if } x \neq 0.$$

Hence if $z \to 0$ along the path $y = mx, f(z) \to \frac{m}{1+m^2}$ which is different for different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$ so that $f(z)$ is not even continuous at $z = 0$.

Thus $f(z)$ is not differentiable at $z = 0$.

Example 2:

Let $f(z) = \sqrt{|xy|}$

Here $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$.

$$u_x(0,0) = \lim_{h\to 0}\left[\frac{u(h,0)-u(0,0)}{h}\right]$$
$$= 0$$

Similarly $u_y(0,0) = 0$.

Also $v_x(0,0) = 0$ and $v_y(0,0) = 0$.

Hence the C.R equations are satisfied at $z = 0$.

We claim that $f(z)$ is not differentiable at $(0,0)$.

Along the path $y = mx$,

$$\frac{f(z) - f(0)}{z} = \frac{\sqrt{|xmx|}}{x + imx} = \frac{\sqrt{|m|}}{1 + im} \text{ if } x \neq 0$$

Hence as $z \to 0$ along the path $y = mx$, $\frac{f(z)-f(0)}{z}$ tends to $\frac{\sqrt{|m|}}{1+im}$ which depend on the path along which $z \to 0$ so that f is not differentiable at $z = 0$.

Note. In this example the function $f(z)$ is continuous and has partial derivatives which satisfy Cauchy-Riemann equations at 0 but is not differentiable at 0.

In the following theorem we prove that C.R equations together with the continuity of partial derivatives give a sufficient condition for differentiability of complex functions.

Theorem 2:

Let $f(z) = u(x, y) + iv(x, y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u, v satisfy the Cauchy-Riemann equations at a point $(x, y) \in D$ then f is differentiable at $z = x + iy$.

Proof:

Since $u(x, y)$ and its first order partial derivatives are continuous at (x, y) we have by the mean value theorem for functions of two variables

$$u(x + h_1, y + h_2) - u(x, y) = h_1 u_x(x, y) + h_2 u_y(x, y) + h_1\varepsilon_1 + h_2\varepsilon_2 \ldots (1)$$

where ε_1 and $\varepsilon_2 \to 0$ as h_1 and $h_2 \to 0$.

Similarly

$$v(x + h_1, y + h_2) - v(x, y) = h_1 v_x(x, y) + h_2 v_y(x, y) + h_1\varepsilon_3 + h_2\varepsilon_4 \ldots. \ (2)$$

where $\varepsilon_3, \varepsilon_4 \to 0$ as h_1 and $h_2 \to 0$.

Let $h = h_1 + ih_2$.

Then $\frac{f(x+h)-f(z)}{h} = \frac{1}{h}[u(x + h_1, y + h_2) - u(x, y)$

$$+jv(x + h_1, y + h_2) - v(x, y) \,|$$

$$= \frac{1}{h}\left[\left(h_1 h_1(x,y) + h_2 h_y(x,y) + h_1\varepsilon_1 + h_2\varepsilon_2\right]\right.$$
$$+i\left\{h_1 v_x(x,y) + h_2 v_y(x,y) + h_1\varepsilon_3 + h_2\varepsilon_4\right)\}$$

using (1) and (2).

$$= \frac{1}{h}\left[h_1\{u_x(x,y) + iv_x(x,y)\} + h_2\{u_y(x,y) + iv_y(x,y)\}\right.$$
$$\left.+h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\varepsilon_4)\right]$$
$$= \frac{1}{h}\left[(h_1 + ih_2)u_x(x,y) - i(h_1 + ih_2)u_y(x,y)\right.$$
$$\left.+h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\varepsilon_4)\right]$$

(using C.R equations).

$$= \frac{1}{h_1}\left[hu_x(x,y) - ihu_y(x,y) + h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\varepsilon_4)\right]$$
$$= u_x(x,y) - iu_y(x,y) + \frac{h_1}{h}(\varepsilon_1 + i\varepsilon_3) + \frac{h_2}{h}(\varepsilon_2 + i\varepsilon_4).$$

Now, since $\left|\frac{h_1}{h_r}\right| \leq 1, \frac{h_1}{h}(\varepsilon_1 + i\varepsilon_3) \to 0$ as $h \to 0$.

Similarly $\frac{h_2}{h}(\varepsilon_2 + i\varepsilon_4) \to 0$ as $h \to 0$.

$\therefore \lim\limits_{h\to 0} \frac{f(z+h)-f(z)}{h} = u_x(x,y) - iu_y(x,y)$. Hence f is differentiable.

Example 1:

Let $f(z) = e^x(\cos y + i\sin y)$.

$\therefore u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

Then $u_x(x,y) = e^x \cos y = v_y(x,y)$

and $u_y(x,y) = -e^x \sin y = -v_x(x,y)$

Thus the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations at every point.

Further $u(x,y)$ and $v(x,y)$ and their first order partial derivatives are continuous at every point. Hence f is differentiable at every point of the complex plane.

Example 2:

Let $f(z) = |z|^2$.

$\therefore f(z) = u(x,y) + iv(x,y) = x^2 + y^2$.

$\therefore u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

Hence $u_x(x,y) = 2x; u_y(x,y) = 2y$

$v_x(x,y) = 2x = v_y(x,y)$.

Clearly the Cauchy-Riemann equations are satisfied at $z \approx 0$.

Further u and v and their first order partial derivatives are continuous and hence f is differentiable at z=0.

Also, we notice that the C.R equations are not satisfied at any point $z \neq 0$ and hence is not differentiable at $z \neq 0$. Thus f is differentiable only at $z = 0$.

Alternate forms of Cauchy - Riemann equations

In the following theorem we express the Cauchy-Riemann equations in complex form.

Theorem 3: (Complex form of C-R equations)

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable. Then the C.R equations can be put in the complex form as $f_x = -if_y$.

Proof:

Let $f(z) = u(x,y) + iv(x,y)$.

Then

$$\begin{aligned} f_x &= u_x + iv_x \\ \text{and } f_y &= u_y + iv_y \\ \text{Hence } f_x &= -if_y \\ &\Leftrightarrow u_x + iv_x = -i(u_y + iv_y) \\ &\Leftrightarrow u_x + iv_x = v_y - iu_y \\ &\Leftrightarrow u_x = v_y \text{ and } v_x = -u_y \end{aligned}$$

Thus, the two C.R equations are equivalent to the equation $f_x = -if_y$.

In the following theorem we express the Cauchy Riemann equations and the derivative of a complex function in terms of it polar coordinates.

Theorem 4: (C.R equations in polar coordinates)

Let $f(z) = u(r,\theta) + iv(r,\theta)$ be differentiable at $z = re^{i\theta} \neq 0$.

Then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Further $f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$.

Proof:

We have $x = r\cos\theta$ and $y = r\sin\theta$.

Hence
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \ \ldots\ldots\ldots.(1)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

Also $\therefore$
$$\frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$$
$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \text{ (using C.R equations)}$$
$$= \frac{\partial u}{\partial r} \quad (using\ 1)$$

Thus $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$. Similarly we can prove that $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right]$$
$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$
$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

Now,
$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$$
$$= xf'(z) + iyf'(z)$$
$$= (x + iy)f'(z)$$
$$= zf'(z)$$
$$\therefore\ f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

We now proceed to express C.R equations in yet another form.

Let $f(z) = u(x,y) + iv(x,y)$.

Since $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ we have

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Thus f can be thought of as a function of z and Ξ. Though z and $\bar{z}$ are not independer variables we form the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ as if z and z are independent variable with this convention we have the following theorem.

Theorem 5:

If $f(z)$ is a differentiable function, the C.R equations can be put in 1 form $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Thus $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ which is the complex form of the $\mathrm{C} - \mathrm{R}$ equations.

(ref theorem 3)

Thus, the C.R equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

Solved Problems

Problem 1:

Verify Cauchy-Riemann equations for the function $f(z) = z^3$.

Solution:

$$f(z) = z^3 = (x+iy)^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$\therefore u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$

$\therefore u_x = 3x^2 - 3y^2$ and $v_x = 6xy$

$u_y = -6xy$ and $v_y = 3x^2 - 3y^2$

Here $u_x = v_y$ and $u_y = -v_x$

Hence the Cauchy-Riemann equations are statisfied.

Problem 2:

Prove that the following functions are nowhere differentiable.

(i) $f(z) = \text{Re}z$

(ii) $f(z) = e^x(\cos y - i\sin y)$

Solution.

(i) $f(z) = \text{Re}z = x$

$\therefore u(x,y) = x$ and $v(x,y) = 0$.

$\therefore u_x = 1$ and $v_x = 0$.

$u_y = 0$ and $v_y = 0$.

Since $u_x \neq v_y$ the C.R equations are not satisfied at any point.

Hence $f(z)$ is nowhere differentiable.

(ii) $f(z) = e^x(\cos y - i\sin y)$

$= e^x\cos y - ie^x\sin y$.

$\therefore u(x,y) = e^x\cos y$ and $v(x,y) = -e^x\sin y$.

$\therefore u_x = e^x\cos y$ and $v_x = -e^x\sin y$.

$u_y = -e^x\sin y$ and $v_y = -e^x\cos y$.

Clearly C.R equations are not satisfied at any point and hence $f(z)$ is nowhere differentiable.

Problem 3:

Prove that $f(z) = \begin{cases} \frac{z\text{Re}z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is continuous at $z = 0$ but not differentiable at $z = 0$.

Solution:

First, we shall prove that $\lim_{z\to 0} f(z) = 0$.

Now $|f(z) - 0| = \left|\frac{z\mathrm{Re}z}{|z|}\right| = |\mathrm{Re}z|$.

Further $|\mathrm{Re}z| \leq |z|$.

$\therefore$ For any given $\varepsilon > 0$ if we choose $\delta = \varepsilon$ we get

$|z| = |z - 0| < \delta \Rightarrow |f(z) - 0| < \varepsilon$.

Hence f is continuous at $z = 0$.

Now, we prove that $f(z)$ is not differentiable at $z = 0$.

$$\frac{f(z) - f(0)}{z - 0} = \frac{z\mathrm{Re}z}{z|z|} = \frac{\mathrm{Re}z}{|z|}$$
$$= \frac{x}{\sqrt{x^2 + y^2}} \text{ where } z = x + iy.$$

Along the path $y = mx$,

$$\frac{f(z) - f(0)}{z - 0} = \frac{x}{\sqrt{x^2 + m^2x^2}} = \frac{1}{\sqrt{1 + m^2}}$$

The value of the limit depends on m and hence on the path along which

$z \to 0$.

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore f(z)$ is not differentiable at $z = 0$.

Problem 4:

Prove that $f(z) = z\mathrm{Im}z$ is differentiable only at $z = 0$ and find $f'(0)$.

Solution:

$f(z) = z\,\mathrm{lm}\,z$

$= (x + iy)y$

$\therefore u(x, y) = xy$ and $v(x, y) = y^2$.

$\therefore u_x = y; v_x = 0; u_y = x$ and $v_y = 2y$.

Clearly the C.R equation are satisfied only at $z = 0$.

Further all the first order partial derivatives are continuous.

Hence $f(z)$ is differentiable at $z = 0$.

Also $f'(0) = u_x(0,0) + iv_x(0,0) = 0$.

Problem 5:

Show that $f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is not differentiable at $z = 0$.

Solution. $\frac{f(z)-f(0)}{z-0} = \frac{xy^2(x+iy)}{x^2+y^4} \cdot \left(\frac{1}{x+iy}\right)$

$$= \frac{xy^2}{x^2 + y^4}$$

$\therefore$ Along the path $x = my^2$.

$$\frac{f(z) - f(0)}{z - 0} = \frac{my^4}{m^2y^4 + y^4} = \frac{m}{m^2 + 1}$$

The value of the limit depends on m and hence depends on the path along which $z \to 0$.

$\therefore \lim\limits_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore f(z)$ is not differentiable at $z = 0$.

Problem 6:

Prove that the function $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ satisfies C-R equations at the origin but $f'(0)$ does not exist.

Solution:

$$f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Here $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$ and $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $u(0,0) = v(0,0) = 0$. $u(0,0) = v(0,0) = 0$.

Now, $u_x(0,0) = \lim_{h \to 0} \dfrac{u(h,0) - u(0,0)}{h}$

$$= \lim_{h \to 0} \left(\frac{h^3/h^2 - 0}{h} \right) = 1$$

Similarly, $u_y(0,0) = -1; v_x(0,0) = 1$ and $v_y(0,0) = 1$. (verify)

Thus $u_x(0,0) = v_y(0,0) = 1$ and

$u_y(0,0) = -v_x(0,0) = -1$, so that

C.R. equations are satisfied at $z = 0$.

Now, $\dfrac{f(z) - f(0)}{z - 0} = \dfrac{x^3 - y^3}{(x^2 + y^2)(x + iy)} + i\dfrac{x^3 + y^3}{(x^2 + y^2)(x + iy)}$

Along the path $y = mx$ we have

$$\frac{f(z) - f(0)}{z - 0} = \frac{x^3 - m^3x^3}{(x^2 + m^2x^2)(x + imx)} + i\frac{x^3 + m^3x^3}{(x^2 + m^2x^2)(x + imx)}$$

$$= \frac{1 - m^3}{(1 + m^2)(1 + im)} + i\frac{1 + m^3}{(1 + m^2)(1 + im)}$$

Hence the value of the limit depends on the path along which $z \to 0$.

Thus $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist.

Hence f is not differentiable at 0.

Problem 7:

Prove that $f(z) = \sin x \cosh y + i\cos x \sinh y$ is differentiable at every point.

Solution:

$f(z) = \sin x\cosh y + i\cos x\sinh y$

$\therefore u(x,y) = \sin x\cosh y$ and $v(x,y) = \cos x\sinh y$

$u_x = \cos x\cosh y$ and $v_x = -\sin x\sinh y$

$u_y = \sin x\sinh y$ and $v_y = \cos x\cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$ for all x, y.

Hence C.R. equations are satisfied at every point.

Further all the first order partial derivatives are continuous.

Hence $f(z)$ is differentiable at every point.

Problem 8:

Find constants a and b so that the function $f(z) = a(x^2 - y^2) + ibxy + c$ is differentiable at every point.

Solution:

Here $u(x,y) = a(x^2 - y^2) + c$ and $v(x,y) = bxy$.

$u_x = 2ax; v_x = b_y$.

$u_y = -2ay$ and $v_y = bx$.

Clearly $u_x = v_y$ and $u_y = -v_x$ iff $2a = b$.

∴ C-R equations are satisfied at all points iff $2a = b$.

∴ The function $f(z)$ is differentiable for all values of a, b with $2a = b$.

Problem 9:

Show that $f(z) = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$ where $r > 0$ and $0 < \theta < 2\pi$ is differentiable and find $f'(z)$.

Solution:

$f(z) = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$.

$u = \sqrt{r}\cos(\theta/2)$ and $v = \sqrt{r}\sin(\theta/2)$.

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos(\theta/2) \text{ and } \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin(\theta/2)$$

$$\frac{\partial u}{\partial \theta} = \frac{-\sqrt{r}}{2}\sin(\theta/2) \text{ and } \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2}\cos(\theta/2)$$

Now,

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\sqrt{r}}{2}\cos(\theta/2)\right)$$

$$= \frac{1}{2\sqrt{r}}\cos(\theta/2)$$

$$= \frac{\partial u}{\partial r}$$

Thus $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

Similarly $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

$$= \frac{1}{2\sqrt{r}}\sin(\theta/2)$$

Hence the C-R equations (in polar form) are satisfied.

Further all the first order partial derivatives are continuous.

Hence $f'(z)$ exists.

$$\begin{aligned}\text{Also } f'(z) &= \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \ (refer \text{ theorem } 4)\\ &= \frac{r}{z}\left(\frac{1}{2\sqrt{r}}\cos(\theta/2) + \frac{i}{2\sqrt{r}}\sin(\theta/2)\right)\\ &= \frac{r}{2\sqrt{r}z}(\cos\theta/2 + i\sin\theta/2)\\ &= \frac{1}{2z}[\sqrt{r}(\cos\theta/2 + i\sin\theta/2)]\\ &= \frac{1}{2z}[\sqrt{z}] = \frac{1}{2\sqrt{z}}\end{aligned}$$

Hence $f'(z) = \frac{1}{2\sqrt{z}}$.

Exercises.

1.Verify C.R. equations for the following functions

(i) $f(z) = az + b$

(ii) $f(z) = e^{z}$

(iii) $f(z) = (1/z), z \neq 0$

(iv) $f(z) = iz + 2$

(v) $f(z) = e^{-x}(\cos y - i\sin y)$

(vi) $f(z) = \cos x\cosh y - i\sin x\sinh$

(vii) $f(z) = \sin z$

(viii) $f(z) = ze^{-z}$

2.Prove that the following are nowhere differentiable.

(i) $f(z) = |z|$

(ii) $f(z) = \operatorname{Im} z$

(iii) $f(z) = xy + iy$

(iv) $f(z) = z - \bar{z}$

(v) $f(z) = 2x + ixy^2$

3.Prove that for the following functions the C.R. equations are satisfied at $z = 0$, but the function is not differentiable at $z = 0$.

(i) $f(z) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

(ii) $f(z) = \begin{cases} \frac{x^2y^5(x+iy)}{x^4+y^{10}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

(iii) $f(z) = \begin{cases} \frac{x^3y(y-ix)}{x^6+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

(iv) $f(z) = \begin{cases} \frac{z^5}{|z^4|} \text{ if } z \neq 0 \\ 0 \text{ if } z = 0 \end{cases}$

4.Prove that the following functions are differentiable at every point.

(i) $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

(ii) $f(z) = iz + 2$

(iii) $f(z) = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$

(iv) $f(z) = 2x - 3y + i(3x + 2y)$.

5.Find constants a, b and c so that the following functions are differentiable at every point.

(i) $f(z) = x + ay - i(bx + cy)$

(ii) $f(z) = x + ay + i(bx + cy)$

(iii) $f(z) = ax^2 - by^2 + icxy$

(iv) $f(z) = e^x \cos ay + ie^x \sin(y + b) + c$

(v) $f(z) = \cos x(\cosh y + a\sinh y) + i\sin x(\cosh y + b\sinh y)$

Answers.

5. (i) $a = b; c = -1$ (ii) $a = -b; c = 1$ (iii) $a = c/2 = b$ (v) $a = -1 = b$.

1.7. Analytic Functions:

Definition:

A function f defined in a region D of the complex plane is said to be analytic at a point $a \in D$ if f is differentiable at every point of some neighbourhood of a. Thus f is analytic at a if there exists $\varepsilon > 0$ such that f is differentiable at every point of the disc $S(a, \varepsilon) = \{z/|z - a| < \varepsilon\}$.

If f is analytic at every point of a region D then f is said to be analytic in D. A function which is analytic at every point of the complex plane is called an entire function or integral function.

For example, any polynomial is an entire function.

Remark 1.

If f is analytic at a point a then f is differentiable at a. However, the converse is not true.

For example, $f(z) = |z|^2$ is differentiable only at $z = 0$. (refer example 2 in theorem 5). Hence f is differentiable at $z = 0$ but not analytic at $z = 0$.

Remark 2.

If $f(z)$ is analytic at a then there exists $\varepsilon > 0$ such that $f(z)$ is differentiable at each point of $S(a, \varepsilon)$. Now, let $z \in S(a, \varepsilon)$. Then we can find $\delta > 0$ such that $S(z, \delta) \subseteq S(a, \varepsilon)$. Hence f is differentiable at every point of $S(z, \delta)$ so that f is analytic at 2.

Thus f is analytic at every point of $S(a, \varepsilon)$. Hence f is analytic at a and if only if f is analytic at each point of some neighbourhood of a. Hence the set of all points for which a given function is analytic forms an open set.

In particular, if a function is analytic in an arbitrary subset A of the complex plane, then there exists an open set containing A in which the function is analytic.

Remark 3.

We shall later prove that if $f(z)$ is analytic at a point then $f(z)$ has derivatives of all orders at that point. In particular

$$f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

is further differentiable and hence $f'(z)$ is continuous.

Hence u_x, v_x, u_y, v_y are all continuous. This together with Theorem 4 gives the following result. $f(z)$ is analytic in a region D if and only if the real and imaginary parts of $f(z)$ have continuous first order partial derivatives that satisfy the Cauchy-Riemanns equations for all points in D.

Further it follows that if $f(z)$ is analytic in D then u and v have continuous partial derivatives of all orders.

Theorem 1:

An analytic function in a region D with its derivative zero at every point of the domain is a constant.

Proof:

Let $f(z) = u(x,y) + iv(x,y)$ be analytic in D and $f'(z) = 0$ for all $z \in D$.

Since $f'(z) = u_x + iv_x = v_y - iu_y$ we have $u_x = u_y = v_x = v_y = 0$.

$\because$ $u(x,y)$ and $v(x,y)$ are constant functions and hence $f(z)$ is constant.

Remark 4.

The above theorem is not true if the domain of $f(z)$ is not a region. For example, let $D = \{z/|z| < 1\} \cup \{z/|z| > 2\}$

D is not a connected subset of C so that D is not a region.

Let $f: D \to C$ be defined by

$$f(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ 2 & \text{if } |z| > 2 \end{cases}$$

Clearly $f'(z) = 0$ for all points $z \in D$ and f is not a constant function in D.

Solved problems

Problem 1:

An analytic function in a region with constant modulus is constant.

Solution:

Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D.

Since $|f(z)|$ is constant, we have $u^2 + v^2 = c$ where c is a constant.

Differentiating partially with respect to x we get $2uu_x + 2vv_x = 0$.

$$\text{(i.e)}\ uu_x + vv_x = 0 \ldots \ldots \ldots . (1)$$

Similarly differentiating partially with respect to y we get

$$uu_y + vv_y = 0 \ldots \ldots \ldots . . (2)$$

Using C.R equations in (1) and (2) we get

$$uu_x - vu_y = 0 \ldots \ldots \ldots . (3)$$
$$uu_y + vu_x = 0 \ldots \ldots \ldots . (4)$$

Eliminating u_y from (3) and (4) we get $(u^2 + v^2)u_x = 0$.

Since $u^2 + v^2 = c$ we get $u_x = 0$.

Similarly we can prove that $v_x = 0$ so that $f'(z) = u_x + iv_x = 0$.

Hence f is constant.

Problem 2:

Any analytic function $f(z) \doteq u + iv$ with $\arg f(z)$ constant is itself a constant function.

Solution:

$\arg f(z) = \tan^{-1}(v/u) = c$, where c is a constant.

$\therefore \frac{v}{u} = k$ where k is a constant.

$\therefore v = ku$.

Hence $v_x = ku_x$ and $v_y = ku_y$.

Eliminating k from the above equations we get $u_x v_y = v_x u_y$.

$\therefore u_x v_y - u_y v_x = 0.$

$\therefore u_x^2 + u_y^2 = 0$ (using C.R. equations)

$\therefore u_x = 0$ and $u_y = 0$ and hence u is constant.

Similarly we can prove that v is constant.

$\therefore f = u + iv$ is constant.

Problem 3:

If $f(z)$ and $\overline{f(z)}$ are analytic in a region D show that $f(z)$ is constant in that region.

Solution:

Let $f(z) = u(x,y) + iv(x,y)$.

$$\begin{aligned}\therefore \overline{f(z)} &= u(x,y) - iv(x,y)\\ &= u(x,y) + i[-v(x,y)]\end{aligned}$$

Since $f(z)$ is analytic in D we have $u_x = v_y$ and $u_y = -v_x$.

Since $\overline{f(z)}$ is analytic in D we have $\|_x = -v_y$ and $u_y = v_x$.

Adding we get $\|_x = 0$ and $u_y = 0$.

Hence $u_x = 0 = v_x$

$\therefore f'(z) = u_x + iv_x = 0$

$\therefore f(z)$ is constant in D.

Problem 4:

Prove that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

Solution:

Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a region D.

Then the first order partial derivatives of u and v are continuous and satisfy the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \dots\dots\dots..(1)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \dots \dots \dots (2)$$

Now, $\overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$.

$= u_1(x, y) + iv_1(x, y)$ where $u_1(x, y) = u(x, -y)$

and $v_1(x, y) = -v(x, -y)$.

Hence $\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v_1}{\partial y}$ (using 1)

and $\frac{\partial u_1}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial v_1}{\partial x}$

$\therefore$ The first order partial derivatives of u_1 and v_1 are continuous and satisfy the CauchyRiemann equations in D.

Hence $\overline{f(\bar{z})}$ is analytic in D.

Similarly if $\overline{f(\bar{z})}$ is analytic in D then $f(z)$ is also analytic in D.

Problem 5:

If $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Solution:

Let $z = x + iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

Hence $$= \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\therefore \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial^2}{\partial y \partial x} + i\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{2i}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\frac{\partial^2}{\partial x \partial y} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\left(i + \frac{1}{i}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Exercises.

1.Prove that an analytic function whose real part is constant is itself a constant.

2.Prove that an analytic function whose imaginary part is constant is itself a constant.

3.If $f = u + iv$ is analytic in a region D and uv is constant in D then prove that f reduces to a constant.

4.If $f = u + iv$ is analytic in a region D and $v = u^2$ in D then prove that f reduces to a constant.

5.Determine the constants a and b in order that the function $f(z) = (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$ should be analytic. Find $f'(z)$.

6.Test whether the following functions are analytic

(i) $z^3 + z$

(ii) $e^x(\cos y + i\sin y)$

(iii) $e^x(\cos y - i\sin y)$

(iv) $e^{-x}(\cos y - i\sin y)$

Answers.

5. $a = -1;\ b = 1; f'(z) = (1 + i)z^2$ 6.(i) yes (ii) yes (iii) no (iv) yes.

1.8. Harmonic Functions

Definition:

Let $u(x,y)$ be a function of two real variables x and y defined in a region $D. u(x,y)$ is said to be a harmonic function if $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and this equation is called Laplace's equation.

Theorem 1:

The real and imaginary parts of an analytic function are harmonic functions.

Proof:

Let $f(z)=u(x,y)+iv(x,y)$ be an analytic function.

Then u and v have continuous partial derivatives of first order which satisfy the C.R equations given by $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

Further $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x}$.

Now
$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}&=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)\\&=\frac{\partial^2 v}{\partial x \partial y}-\frac{\partial^2 v}{\partial y \partial x}\\&=0\end{aligned}$$

Thus u is a harmonic function.

Similarly we can prove that v is a harmonic function.

Remark 1. Laplace's equation provides a necessary condition for a function to be the real or imaginary part of an analytic function.

For example, if $u(x,y)=x^2+y$ we have

$$\frac{\partial^2 u}{\partial x^2}=2; \frac{\partial^2 u}{\partial y^2}=0 \text{ and } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2$$

Thus $u(x,y)$ is not harmonic function and hence it cannot be the real part of any analytic function.

Definition:

Let $f = u + iv$ be an analytic function in a region D. Then v is said to be a conjugate harmonic function of 1. .

Theorem 2:

Let $f = || + iv$ be an analytic function in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

Proof:

Let v be a harmonic conjugate of u. Then $f = u + iv$ is analytic.

$\therefore\ if = iu - v$ is also analytic.

Hence u is a harmonic conjugate of $-v$.

The proof for the converse is similar.

Theorem 3:

Any two harmonic conjugates of a given harmonic function u in a region O differ by a real constant.

Proof:

Let u be a harmonic function.

Let v and v^* be two harmonic conjugates of $||$.

$u + iv$ and $u + iv^*$ are analytic in D.

Hence by the Cauchy-Riemann equation we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y}$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v^*}{\partial x}$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial v^*}{\partial x}$$

Hence $\frac{\partial}{\partial y}(v - v^*) = 0$ and $\frac{\partial}{\partial x}(v - v^*) = 0$.

$\therefore v = v^* + c$ where c is a real constant.

Remark:

The Cauchy-Riemann equations can be used to obtain a harmonic conjugate of a given harmonic function.

For example, let $u(x,y) = x^2 - y^2$.

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ so that u is harmonic in the whole complex plane C .

Not, let $v(x,y)$ be a harmonic conjugate of u.

$$\text{Then } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \ldots \ldots . . (1)$$

$$\text{and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y \ldots \ldots \ldots (2)$$

On integration of (1) with respect to y we get $v = 2xy + \varphi(x)$ where $\varphi(x)$ is a function of x alone.

Now from (2) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ gives $2y + \varphi'(x) = 2y$

$\therefore \varphi'(x) = 0$ so that $\varphi(x) = c($ a constant $)$.

$\therefore v = 2xy + c.$

Thus the harmonic conjugate of $u(x,y) = x^2 - y^2$ is given by $v(x,y) = 2xy + c$ and the corresponding entire function is given by

$$\begin{aligned} f(z) &= (x^2 - y^2) + i(2xy + c) \\ &= z^2 + ic \end{aligned}$$

Let $u(x,y)$ and $v(x,y)$ be given harmonic functions. We now describe a method, due to Milne-Thompson, of constructing an analytic function whose real part is $u(x,y)$ or imaginary part is $v(x,y)$.

Milne-Thompson method

Let $u(x,y)$ be a given harmonic function.

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

$$\begin{aligned} \text{Then } f'(z) &= u_x(x,y) + iv_x(x,y) \\ &= u_x(x,y) - iu_y(x,y) \end{aligned}$$

Let $\varphi_1(x, y) = u_x(x, y)$ and $\varphi_2(x, y) = u_y(x, y)$.

We have $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

Hence $f'(z) = \varphi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\varphi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$.

Putting $z = \bar{z}$ we obtain $f'(z) = \varphi_1(z, 0) - i\varphi_2(z, 0)$.

Hence $f(z) = \int [\varphi_1(z, 0) - i\varphi_2(z, 0)]dz + c$.

Note. It can be proved in a similar way that the analytic function $f(z)$ with a given harmonic function $v(x, y)$ as imaginary part is given by

$$f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]dz + c$$

where $\psi_1(x, y) = v_y$ and $\psi_2(x, y) = v_x$.

Solved Problems

Problem 1:

Prove that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

Solution:

$u = 2x - x^3 + 3xy^2$.

$\therefore u_x = 2 - 3x^2 + 3y^2; u_{xx} = -6x; u_y = 6xy; u_{yy} = 6x$.

$\therefore u_{xx} + u_{yy} = 0$. Hence u is harmonic.

Let v be a harmonic conjugate of u.

$\therefore f(z) = u + iv$ is analytic.

By Cauchy-Riemann equations we have

$$v_y = u_x = 2 - 3x^2 + 3y^2.$$

$\therefore$ Integrating with respect to y we get

$$v = 2y - 3x^2y + y^3 + \lambda(x) \dots\dots\dots\dots\dots (1)$$

where $\lambda(x)$ is an arbitrary function of x.

$\therefore v_x = -6xy + \lambda'(x)$.

Now $v_x = -u_y$ gives $-6xy + \lambda'(x) = -6xy$.

Hence $\lambda'(x) = 0$ so that $\lambda(x) = c$ where c is a constant.

Thus $v = 2y - 3x^2y + y^3 + c$ [from (1)].

$$\begin{aligned} \text{Now } f(z) &= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3) + ic \\ &= 2(x+iy) - [(x^3 - 3xy^2) + i(3x^2y - y^3)] + ic \\ &= 2z - z^3 + ic. \end{aligned}$$

$\therefore f(z) = 2z - z^3 + ic$ is the required analytic function.

Problem 2:

Show that $u = \log\sqrt{x^2+y^2}$ is harmonic and determine its conjugate and hence find the corresponding analytic function $f(z)$.

Solution:

$$u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2).$$

$$\therefore u_x = \frac{x}{x^2+y^2}; u_{xx} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Similarly $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$.

Obviously $u_{xx} + u_{yy} = 0$ and hence u is harmonic.

Let v be a harmonic conjugate of u.

$\therefore f(z) = u + iv$ is an analytic function.

By C.R. equations we have

$$v_y = u_x = \frac{x}{x^2+y^2}.$$

Integrating w.r.t y we get $v = \tan^{-1}\left(\frac{y}{x}\right) + \phi(x)$ where $\phi(x)$ is an arbitrary function of x.

Now $v_x = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) + \phi'(x)$.

Also $v_x = -u_y \Rightarrow \frac{-y}{x^2+y^2} + \phi'(x) = \frac{-y}{x^2+y^2}$ so that $\phi'(x) = 0$.

Hence $\phi(x) = c$.

$$\therefore v = \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$\therefore\ f(z) = u + iv = \log\sqrt{x^2+y^2} + i\left[\tan^{-1}\left(\frac{y}{x}\right) + c\right].$$

Problem 3:

Show that $u(x,y) = \sin x\cosh y + 2\cos x\sinh y + x^2 - y^2 + 4xy$

is harmonic. Find an analytic function $f(z)$ in terms of z with the given u for its real part.

Solution:

$$\begin{aligned} u_x &= \cos x\cosh y - 2\sin x\sinh y + 2x + 4y. \\ u_{xx} &= -\sin x\cosh y - 2\cos x\sinh y + 2. \\ u_y &= \sin x\sinh y + 2\cos x\cosh y - 2y + 4x. \\ u_{yy} &= \sin x\cosh y + 2\cos x\sinh y - 2. \end{aligned}$$

$\therefore u_{xx} + u_{yy} = 0$. Hence u is harmonic.

Now let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$.

$$\begin{aligned} \therefore \phi_1(z,0) &= \cos z\cosh 0 - 2\sin z\sinh 0 + 2z \\ &= \cos z + 2z. \end{aligned}$$

Similarly, $\phi_2(z,0) = 2\cos z + 4z$.

$$\begin{aligned} \therefore f(z) &= \int [\phi_1(z,0) - i\phi_2(z;0)]dz(\text{ by Milne Thompson method }) \\ &= \int [\cos z + 2z - i(2\cos z + 4z)]dz \\ &= \sin z + z^2 - 2i\sin z - 2iz^2 + c. \end{aligned}$$

Problem 4:

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function and

$u(x,y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.

Solution:

It can be verified that $u(x,y)$ is harmonic.

Now, $u_x = \frac{(\cosh 2y + \cos 2x)2\cos 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$

$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$.

Also, $u_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$.

Let $\phi_1(x, y) = u_x$ and $\phi_2(x, y) = u_y$

$$\therefore \phi_1(z, 0) = \frac{2\cos 2z \cosh 0 + 2}{(\cosh 0 + \cos 2z)^2} = \frac{2}{1 + \cos 2z} = \sec^2 z$$

and $\phi_2(z, 0) = 0$.

$$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)] dz$$

Now, $$= \int \sec^2 z dz$$

$$= \tan z + c.$$

$$\therefore f(z) = \tan z + c.$$

Problem 5:

Find the analytic function $f(z) = u + iv$ if $u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$.

Solution:

$$u + v = \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad \ldots \ldots (1)$$

$$\therefore u_x + v_x = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} \quad \ldots \ldots \ldots \ldots \ldots (2)$$

$$\text{and } u_y + v_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \quad \ldots \ldots \ldots \ldots (3)$$

Since the required function $f(z) = u + iv$ is to be analytic, u and v satisfy the C.R. equations $u_x = v_y$ and $u_y = -v_x$.

Using these equations in (2) we get

$$u_x - u_y = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\therefore u_x(z,0) - u_y(z,0) = \frac{2(1-\cos 2z)\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{2\sin^2 z} = -\mathrm{cosec}^2 z \quad \ldots\ldots\ldots\ldots(4)$$

Using C.R. equations in (3) we get

$$u_y + u_x = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\therefore\ u_y(z,0) + u_x(z,0) = 0$$

Now adding (4) and (5) we get $2u_x(z,0) = -\mathrm{cosec}^2 z$.

$$\therefore\ u_x(z,0) = -\frac{1}{2}\mathrm{cosec}^2 z \ldots\ldots\ldots\ldots(6)$$

Subtracting (4) from (5) we get $2u_y(z,0) = \mathrm{cosec}^2 z$.

$$\therefore\ u_y(z,0) = \frac{1}{2}\mathrm{cosec}^2 z \ldots\ldots\ldots\ldots(7)$$

Now $f(z) = u(z,0) + iv(z,0)$

$$\Rightarrow f'(z) = u_x(z,0) + iv_x(z,0)$$

$$= u_x(z,0) - iu_y(z,0)$$

$$= -\frac{1}{2}(1+i)\mathrm{cosec}^2 z \qquad [\text{ using (6) and (7)}]$$

Integrating w.r.t z we have

$$f(z) = \left(\frac{1+i}{2}\right)\cot z + c$$

Problem 6:

Given $v(x,y) = x^4 - 6x^2y^2 + y^4$ find $f(z) = u(x,y) + iv(x,y)$ such that $f(z)$ is analytic.

Solution:

It can be easily verified that $v(x,y)$ is harmonic.

Now, $v_x = 4x^3 - 12xy^2$ and $v_y = -12x^2y + 4y^3$.

Let $f(z) = u + iv$ be the required analytic function.

By Cauchy-Riemann equations $u_x = v_y$.

$\therefore u_x = -12x^2y + 4y^3$.

$\therefore$ Integrating with respect to x we get $u = -4x^3y + 4xy^3 + \lambda(y)$ where $\lambda(y)$ is an arbitrary function of y.

$$\therefore u_y = -4x^3 + 12xy^2 + \lambda'(y) = -v_x.$$
$$\therefore -(4x^3 - 12xy^2) = -4x^3 + 12xy^2 + \lambda'(y).$$
$\therefore \lambda'(y) = 0$ so that $\lambda(y) = c$ where c is a constant.

Thus $u = -4x^3y + 4xy^3 + c$.

$$\begin{aligned}\therefore f(z) &= (-4x^3y + 4xy^3 + c) + i(x^4 - 6x^2y^2 + y^4)\\ &= i[(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)] + c\\ &= i(x+iy)^4 + c\\ &= iz^4 + c.\end{aligned}$$

Aliter (Milne Thompson Method).

Let $\psi_1(x,y) = v_y$ and $\psi_2(x,y) = v_x$.

$\therefore \psi_1(x,y) = -12x^2y + 4y^3$ and $\psi_2(x,y) = 4x^3 - 12xy^2$.

$\therefore \psi_1(z,0) = 0$ and $\psi_2(z,0) = 4z^3$.

$$\begin{aligned}\therefore f(z) &= \int [\psi_1(z,0) + i\psi_2(z,0)]dz\\ &= i\int 4z^3dz\\ &= iz^4 + c.\end{aligned}$$

Problem 7:

Find the analytic function $f(z) = u + iv$ given that

$u - v = e^x(\cos y - \sin y)$.

Solution:

$$u - v = e^x(\cos y - \sin y) \quad \ldots\ldots\ldots (1)$$

$$\therefore u_x - v_x = e^x(\cos y - \sin y) \ldots \ldots . . (2)$$
$$\text{and } u_y - v_y = -e^x(\sin y + \cos y) \ldots \ldots \ldots (3)$$

Since the required function is to be analytic it has to satisfy the C.R. equations.

$\therefore$ Using C.R. equations in (3) we get

$$-v_x - u_x = -e^x(\sin y + \cos y) \ldots \ldots \ldots (4)$$

Solving (2) and (4) we get

$$u_x = e^x \cos y \ldots \ldots \ldots \ldots . . (5)$$
$$\text{and } v_x = e^x \sin y \ldots \ldots \ldots . (6)$$

Integrating (6) with respect to x we get

$$v = e^x \sin y + f(y)$$

$$\therefore v_y = e^x \cos y + f'(y) \ldots \ldots \ldots . . (7)$$

Using C.R. equations in (5) and (7) we get $f'(y) = 0$.

Hence $f(y) = c_1$ where c_1 is a constant.

$\therefore v = e^x \sin y + c_1$.

From (1) $u = e^x \cos y + c_2$.

Now,
$$\begin{aligned} f(z) &= u + iv \\ &= e^x(\cos y + ie^x \sin y + c_1 + ic_2 \\ &= e^x(\cos y + i\sin y) + (c_1 + ic_2) \\ &= e^x e^{iy} + \alpha (\text{ where } \alpha \text{ is a complex constant }) \\ &= e^{x+iy} + \alpha \\ &= e^z + \alpha \end{aligned}$$

Problem 8:

If $u + v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ find the analytic function $f(z)$ in terms of z.

Solution:

$u + v = (x - y)(x^2 + 4xy + y^2)$ …………. (1)

Differentiating (1) partially w.r.t x we get

$$u_x + v_x = (x^2 + 4xy + y^2) + (x - y)(2x + 4y) \ldots \ldots \ldots . (2)$$

Differentiating (1) partially w.r.t y we get

$$u_y + v_y = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y) \dots\dots\dots.(3)$$

Since $f = u + iv$ is analytic, u and v satisfy the C.R. equations

$$u_x = v_y \text{ and } u_y = -v_x.$$

$\therefore$ Using C.R. equations in (3) we get

$$-v_x + u_x = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y) \dots\dots\dots(4)$$

Adding (2) and (4) we get $2u_x = (x - y)(6x + 6y)$.

$$\therefore \; u_x = 3(x^2 - y^2) \dots\dots\dots\dots.(5)$$

Subtracting (4) from (2) we get $v_x = 6xy$.

Using C.R. equations in (6) we get $u_y = -6xy$.

Let $\phi_1(x, y) = u_x$ and $\phi_2(x, y) = u_y$.

$\therefore \phi_1(z, 0) = 3z^2$ and $\phi_2(z, 0) = 0$.

By Milne-Thompson method

$$f(z) = \int \; [\phi_1(z, 0) - i\phi_2(z, 0)]dz = \int \; 3z^2 dz = z^3 + c$$

Problem 9:

Find the real part of the analytic function whose imaginary part is

$e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$. Construct the analytic function.

Solution:

Let $v = e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$ and $f(z) = u + iv$ be the required analytic function.

We can prove that v is harmonic. We use Milne Thompson method to find the harmonic conjugate u of v.

Let $\psi_1(x, y) = v_y$

$$= e^{-x}(2x\cos y - 2xy\sin y + 2y\sin y + (y^2 - x^2)\cos y)$$

and $\psi_2(x, y) = v_x$

$= e^{-x}(-2xy\cos y - (y^2 - x^2)\sin y + 2y\cos y - 2x\sin y)$.

$\therefore \psi_1(z, 0) = e^{-2}(2z - z^2)$ and $\psi_2(z, 0) = 0$.

By Milne Thompson method

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]dz$$
$$= \int e^{-z}(2z - z^2)dz$$
$$= \int 2ze^{-z}dz - \left[-z^2e^{-z} + \int e^{-z}2zdz\right]$$
$$= z^2e^{-z}$$
$$= (x+iy)^2e^{-(x+iy)}$$
$$= [(x^2-y^2) + 2ixy]e^{-x}(\cos y - i\sin y)$$
$$= e^{-x}[(x^2-y^2) + 2ixy](\cos y - i\sin y)$$

Real part of $f(z) = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]$.

(i.e.) $u = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]$.

Problem 10:

Find the constant a so that $u(x,y) = ax^2 - y^2 + xy$ is harmonic. Find an analytic function $f(z)$ for which u is the real part. Also find its harmonic conjugate.

Solution:

$u = ax^2 - y^2 + xy.$

Given that u is harmonic. Hence it satisfies Laplace's equation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

$$\frac{\partial u}{\partial x} = 2ax + y \text{ and } \frac{\partial^2 u}{\partial x^2} = 2a$$

Now, $$\frac{\partial u}{\partial y} = -2y + x \text{ and } \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow 2a - 2 = 0$$

Hence $a = 1$.

$\therefore u = x^2 - y^2 + xy.$

Hence $u_x = 2x + y$ and $u_y = -2y + x$.

Let $\phi_1(x, y) = u_x = 2x + y$ and $\phi_2(x, y) = u_y = -2y + x$.

$$\therefore \phi_1(z, 0) = 2z \text{ and } \phi_2(z, 0) = z.$$

$$\begin{aligned}\therefore f(z) &= \int [\phi_1(z, 0) - i\phi_2(z, 0)]dz \\ &= \int (2z - iz)dz \\ &= z^2 - \frac{iz^2}{2} + c \\ &= (x + iy)^2 - i\frac{(x + iy)^2}{2} + c \\ &= (x^2 - y^2 + 2ixy) - \frac{i}{2}(x^2 - y^2 + 2ixy) + c \\ &= (x^2 - y^2 + xy) + i\left(2xy + \frac{y^2 - x^2}{2}\right) + c\end{aligned}$$

$\therefore v(x, y) = 2xy + \left(\frac{y^2 - x^2}{2}\right)$ is the harmonic conjugate of $u(x, y)$.

Problem 11:

If $u(x, y)$ is a harmonic function in a region D prove that $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in D.

Solution. Let $U = \frac{\partial u}{\partial x}$ and $V = -\frac{\partial u}{\partial y}$.

$\therefore f(z) = U + iV$. Since u is harmonic U and V have continuous first order partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Also $\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ [using (1)] $= \frac{\partial V}{\partial y}$.

Hence $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$.

Now, $\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial V}{\partial x}$.

Hence $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$.

Thus the partial derivatives of U and V satisfy the Cauchy-Riemann equations.

Hence f is analytic in D.

Problem 12:

If u and v are harmonic functions satisfying the Cauchy-Riemann equations in a region D then $f = u + iv$ is analytic in D.

Solution:

Since u and v are harmonic the first order partial derivatives of u and v are continuous. Also u and v satisfy the C.R. equations in D.

Hence $f = u + iv$ is analytic in D.

Problem 13:

Prove that the real (imaginary) part of an analytic function when expressed in polar form satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(This equation is the Laplace equation in polar form.)

Solution:

We know that Cauchy-Riemann equations in polar form are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \dots\dots\dots.(1)$$

and

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \dots\dots\dots.(2)$$

We eliminate v from (1) and (2).

Differentiating (1) partially with respect to r and (2) partially with respect to θ we have

$$\frac{\partial^2 v}{\partial r \partial \theta} = r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \dots\dots\dots..(3)$$

$$\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \dots\dots\dots\dots(4)$$

Since $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$ we have $r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$.

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$

Similarly, $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$

Problem 14:

ϕ and ψ are functions of x and y satisfying Laplace's equation.

If $u = \psi_y - \psi_x$ and $v = \phi_x + \psi_y$ prove that $u + iv$ is analytic.

Solution:

Given that ϕ and ψ satisfy Laplace's equation.

Hence $\phi_{xx} + \phi_{yy} = 0$

and $\psi_{xx} + \psi_{yy} = 0.$

Also $u = \phi_y - \psi_x$ and $v = \phi_x + \psi_y$.

Hence $\begin{aligned} u_x &= \phi_{xy} - \psi_{xx} \\ u_y &= \phi_{yy} - \psi_{yx} \end{aligned}$

$$\begin{aligned} v_x &= \phi_{xx} + \psi_{xy} \\ &= -\phi_{yx} + \psi_{xy}[b_y(1)] \\ &= \phi_{yx} + \psi_{yy} \\ &= \phi_{yx} - \psi_{xx}[by(2)]. \end{aligned}$$

and

Thus $u_x = v_y$ and $u_y = -v_x$.

Since φ and ψ are harmonic, all the partial derivatives are continuous.

Hence $u + iv$ is analytic.

Problem 15:

Show that if u and v are conjugate harmonic functions the product uv is a harmonic function.

Solution:

Since u and v are conjugate harmonic functions, we have

$$u_{xx} + u_{yy} \ = 0 \ldots \ldots \ldots (1)$$
$$v_{xx} + v_{yy} \ = 0 \ldots \ldots \ldots .. (2)$$
$$u_x \ = v_y \ldots \ldots \ldots \ldots \ldots . (3)$$
$$u_y \ = -v_x \ldots \ldots \ldots \ldots . (4)$$

Now let $\phi = tv$.

$$\phi_x = uv_x + vu_x$$
$$\phi_{xx} = uv_{xx} + 2u_x v_x + vu_{xx}$$
$$\text{Similarly } \phi_{yy} = uv_{yy} + 2u_y v_y + vu_{yy}$$
$$= uv_{yy} - 2v_x u_x + vu_{yy} [\text{using}(3) \text{ and } (4)\ ]$$
$$\text{Now } \phi_{xx} + \phi_{yy} = u(v_{xx} + v_{yy}) + v(u_{xx} + u_{yy})$$
$$= 0 [\text{using}(1) \text{ and } (2)\ ]$$

$\therefore \phi = uv$ is a harmonic function.

Problem 16:

If $f(z) = u + iv$ is analytic and $f(z) \neq 0$, prove that

(i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f(z)| = 0.$

(ii) $\nabla^2 \text{amp} f(z) = 0.$

Solution:

$\log f(z) = \log |f(z)| + i \text{amp} f(z).$

Since $f(z) \neq 0, \log |f(z)|$ exists.

Further since $f(z)$ is analytic and $f(z) \neq 0, \log f(z)$ is also analytic.

$\therefore$ $\log |f(z)|$ and amp $f(z)$ are the real and imaginary parts of the analytic function $\log f(z)$.

Hence both $\log |f(z)|$ and amp $f(z)$ satisfy the Laplace equation.

(i) $\frac{\partial^2}{\partial x^2}(\log |f(z)|) + \frac{\partial^2}{\partial y^2}(\log |f(z)|) = 0$

(i.e.) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f(z)| = 0.$

(ii) Also, $\frac{\partial^2}{\partial x^2}(\text{amp} f(z)) + \frac{\partial^2}{\partial y^2}(\text{amp} f(z)) = 0$

$$\text{(i.e.)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \text{amp } f(z) = 0$$

$$\text{(i.e.) } \nabla^2 \text{ amp } f(z) = 0$$

Problem 17:

Given the function $w = z^3$ where $w = u + iv$. Show that u and v satisfy the Cauchy-Riemann equations. Prove that the families of curves $u = c_1$ and $v = c_2$(c_1 and c_2 are constants) are orthogonal to each other.

Solution:

$w = z^3 = (x + iy)^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$.

$\therefore u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ and $u_y = -6xy$;

$v_x = 6xy$ and $v_y = 3x^2 - 3y^2$.

We note that $u_x = v_y$ and $u_y = -v_x$.

Hence u and v satisfy the Cauchy-Riemann equations.

Now $u_{xx} = 6x$ and $u_{yy} = -6x$.

$\therefore u_{xx} + u_{yy} = 6x - 6x = 0$.

Hence u satisfies the Laplace equations.

Similarly $v_{xx} + v_{yy} = 6y - 6y = 0$.

Hence v satisfies the Laplace equations.

$u = c_1 \Rightarrow x^3 - 3xy^2 = c_1$

Differentiating w.r.t x we get $3x^2 - 3\left(2xy\frac{dy}{dx} + y^2\right) = 0$.

$$\therefore \frac{dy}{dx} = \frac{3(x^2 - y^2)}{6xy} = \frac{x^2 - y^2}{2xy}.$$

$\therefore$ Slope of the tangent at (x_0, y_0) for the curve $u = c_1$ is given by

$m_1 = \frac{x_0^2 - y_0^2}{2x_0 y_0}$.

Now $v = c_2 \Rightarrow 3x^2y - y^3 = c_2$.

Differentiating w.r.t x we get

$3\left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$. Hence $\frac{dy}{dx}(3x^2 - 3y^2) = -6xy$

$$\therefore \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2}$$

Slope of the tangent to the curve $u = c_2$ at (x_0, y_0) is given by $m_2 = \frac{-2x_0 y_0}{x_0^2 - y_0^2}$.

Clearly, $m_1 m_2 = -1$.

.. The two families of curves are orthogonal.

Exercises.

(1) Prove that the following functions are harmonic. Also find a harmonic conjugate.

(i) $u = \sinh x \sin y \Rightarrow\ ux$ wkT. $u_x = v_y$

(ii) $u = 3x^2y + 2x^2 - y^3 - 2y^2$ (iii) $u = e^x \cos y$.

(2) Prove that the following functions are harmonic. Also find a function v such the $f(z) = u + iv$ is analytic and express $f(z)$ in terms of z.

(i) $u = 2x(1 - y)$ (ii) $u = e^x(x\cos y - y\sin y)$

(iii) $u = 2xy + 3y$ (iv) $u = \frac{y}{x^2 + y^2}$.

3. Find the function $f(z) = u + iv$ such that $f(z)$ is analytic given that

(i) $u = x$ (ii) $u = e^x \cos y$ (iii) $u = x^3 - 3xy^2$

(iv) $u = e^x \sin y$ (v) $u = \cos x \cosh y$

(vi) $u = e^x(x\cos y - y\sin y)$ (vii) $u = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$

(viii) $v = 3x^2y - y^3$ (ix) $v = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

(x) $u = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$.

4. Find the analytic function $f(z) = u + iv$ if

(i) $u - v = (x - y)(x^2 + 4xy + y^2)$

(ii) $u + v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$

(iii) $u - v = \frac{\cos x+\sin x-e^{-y}}{2\cos x-e^{y}-e^{-y}}$ given that $f(\pi/2)$

(iv) $u + v = \frac{x}{x^2+y^2}$ given that $f(1) = 1$.

5. If $f(z) = u + iv$ is an analytic function prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2.$$

6.If $f(z)$ is an analytic function of z show that

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2 = |f'(z)|^2.$$

7.Prove that the function $u(z)$ and $u(z)$ are simultancously harmonic.

8.Prove that the function $u(x, y)$ and $u(x^2 - y^2, 2xy)$ are simultaneously harmonic.

9.Prove that $u(x, y) = x^2 - y^2$ and $v(x, y) = -\frac{y}{x^2+y^2}$ are both harmonic but $u + iv$ is not analytic.

10.From the Laplace's equation for $u(x, y)$ prove that $\frac{\partial^2 u}{\partial z \partial z} = 0$.

Answers.

1.(i) $v = -\cosh x \cos y$ (ii) $v = 4xy - x^3 + 3xy^2$ (iii) $v = e^x \sin y$

2.(i) $2z + iz^2$ (ii) ze^z (iii) $-i(z^2 + 3z)$ (iv) $1/z$

3.(i) z (ii) e^z (iii) z^3 (iv) $-ie^z$ (v) $\cos z$ (vi) ze^z (vii) $\sec z$ (viii) z^3 (ix) $i(z^3 + 3z)(x)\cot z$ 4.(ii) $\cot z$ (iii) $\frac{1}{2}(1 - \cot(z/2))$ (iv) $\frac{1+i}{2z} + \frac{1-i}{2}$

UNIT II

Conformal Mapping–Elementary Transformation –Bilinear Transformation – Cross Ratio –Fixed Points.

Chapter 2: Section 2.1 to 2.5

2.1. Conformal Mapping:

In this section we study the geometric consequences of the existence of the derivatives of complex functions. In particular we prove that if an analytic function f has a non zero derivative at a point z_0 lying in a region, then f preserves the angle between any two curves at z_0 both in magnitude and direction.

Definition:

A curve C in the complex plane is given by a continuous function $\gamma:[a,b] \to C$. If $\gamma(t) = x(t) + iy(t)$ then the curve C is determined by the two continuous real valued functions of the real parameter t given by

$x = x(t)$ and $y = y(t)$ where $a \leq t \leq b$.

We also write $z = z(t) = x(t) + iy(t)$ where $a \leq t \leq b$.

The point $z(a)$ is called the origin of the curve and $z(b)$ is called the terminus of the curve.

The curve C is said to be simple if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$. Equivalently C is simple if the function γ is $1-1$.

The curve C is called a closed curve if $z(a) = z(b)$ and C is called a simple closed curve if (i) $z(a) = z(b)$ (ii) $z(t_1) \neq z(t_2)$ for any other pair of distinct real numbers $t_1, t_2 \in [a,b]$.

A simple closed curve is also called a Jordan curve.

A curve C is said to be differentiable if $z'(t)$ exists and is continuous. If further $z'(t) \neq 0$ then the curve is said to be regular (smooth).

Geometrically the regular curve has a tangent whose direction is determined by the argument of $z'(t)$.

If C is a curve determined by the equation $z = z(t)$ where $a \le t \le b$ then the opposis curve of C denoted by $-C$ is given by the equation

$z(t) = z(b + a - t)$ where $a \le t$

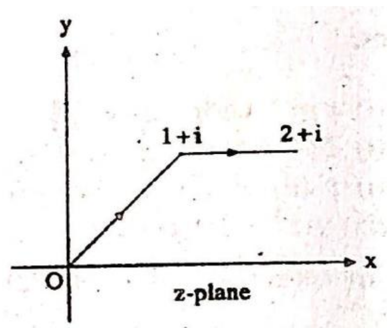
Example 1:

The polygonal line given by $z(t) = \begin{cases} t + it & \text{if } 0 \le t \le 1 \\ t + i & \text{if } 1 \le t \le 2 \end{cases}$

consisting of a line segment from 0 to $1 + i$ followed by another line segment from $1 + i$ to $2 + i$ is a simple curve.

The equation of the curve automatically determines an orientation for the curve as shown in the figure.

We notice that the above curve is differentiable except at $1 + i$. Such a curve is called a piecewise differentiable curve.



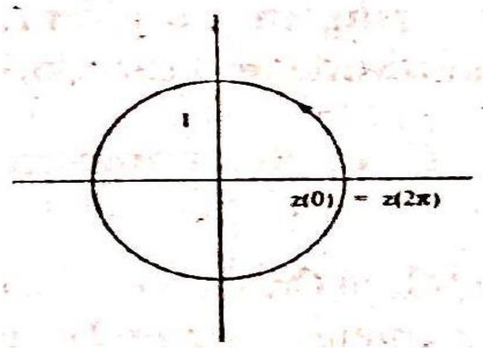
Definition:

A curve C given by $z = z(t)$ is said to be piecewise differentiable if it is differentiable except at a finite number of points and at any point where $z(t)$ is not differentiable it has a left derivative and right derivative.

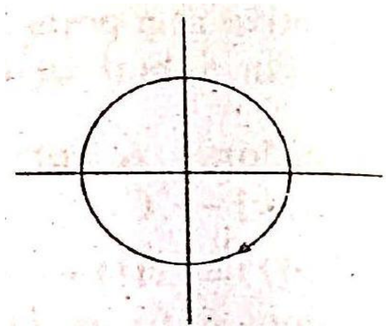
Example 2.

The equation given by $z(t) = \cos t + i\sin t$ where $0 \le t \le 2\pi$ represents the unit circle C with centre O and radius 1 described in the anticlockwise direction.

The origin and terminus of the curve are $z(0) = 1 = z(2\pi)$. The orientation of the circle as described in the figure is taken as the positive orientation.



Positive Orientation



Negative Orientation

The same circle with negative orientation $-C$ is given by the equation $z(t) = \cos(2\pi - t) + i\sin(2\pi - t)$. This is a simple closed curve.

Example 3:

In general, the equation $z(t) = a + r(\cos t + i\sin t)$ where $0 \le t \le 2\pi$ represents a positively oriented circle with centre a and radius r. This is also a simple closed curve.

Example 4:

The curve represented by $z(t) = \cos t + i\sin t$ where $0 \le t \le 4\pi$ is a closed curve. However, it is not a simple closed curve, since $z(\pi/2) = z(5\pi/2)$. Actually, the equation represents a unit circle traversed twice.

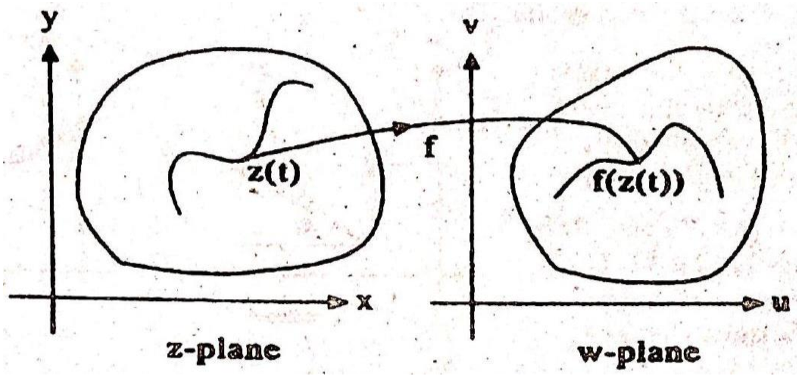
Example 5:

The curve represented by $z(t) = \cos t + i\sin t$ where $0 \le t \le \pi$ represents a semi-circular curve of unit radius above the real axis with the origin 1 and terminus -1. This is not a closed curve since $z(0) \neq z(\pi)$.

Definition:

Let f be an analytic function in a region D. Let C be a curve given by the equation $z = z(t)$ where $a \le t \le b$ and lying in D.

Then the equation $w = w(t) = f(z(t))$ defines another curve C' in the w-plane and is called the image of the curve C under f.



Definition:

Let f be a continuous function defined in the region D. Let $z_0 \in D$. Let C_1 and C_2 be two regular curves passing through z_0 and lying in D. Let C_1' and C_2' be the images of C_1 and C_2 respectively under f. If the angle between C_1 and C_2 is equal to the angle between C_1' and C_2' both in magnitude and direction then f is said to be conformal at z_0.

Thus, a conformal mapping preserves angle both in magnitude and direction.

If the angle is preserved only in magnitude and direction is reversed then the mapping is said to be isogonal or indirectly conformal.

Theorem 1:

Let f be an analytic function defined in a region D. Let $z_0 \in D$. If $f'(z_0) \neq 0$ then f is conformal at z_0.

Proof:

Let C be a regular curve lying in the region D and passing thorugh $z_0 \in D$.

Suppose the equation C is given by $z = z(t)$ where $a \leq t \leq b$.

Let $z_0 = z(t_0)$ for $t_0 \in [a, b]$.

The equation of the image curve C' of C under f is given by $w = w(t) = f(z(t))$.

$\therefore w'(t) = f'(z(t))z'(t).$

$\therefore w'(t_0) = f'\big(z(t_0)\big)z'(t_0)$

$= f'(z_0)z'(t_0).$

By hypothesis $f'(z_0) \neq 0$.

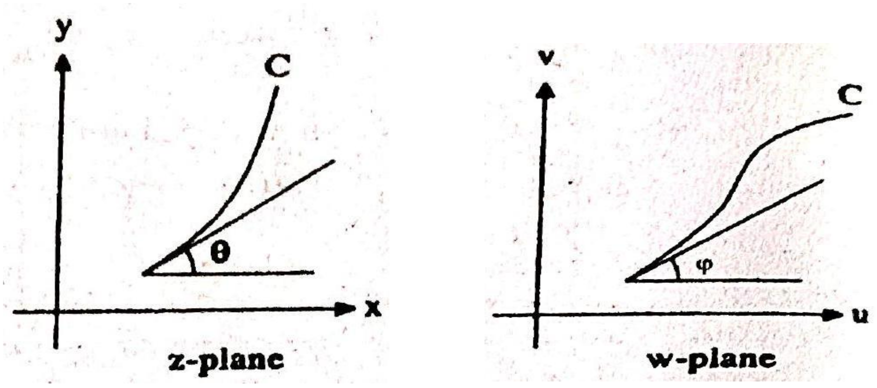
Also since C is regular $z'(t_0) \neq 0$.

Hence $w'(t_0) \neq 0$ and $\arg w'(t_0) = \arg f'\big(z(t_0)\big) + \arg z'(t_0)$ ....(1)

Hence $\varphi = \psi + \theta$ where $\quad \arg w'(t_0) = \varphi$, $\arg f'(z_0) = \psi$ and $\arg z'(t_0) = \theta$.

Clearly θ represents the angle made by the tangent to the curve C at z_0 with the positive direction of the x-axis in the z-plane.

Similarly φ represents the angle made by the tangent to the curve C' at $f(z_0)$ with the positive direction of the u-axis in the w-plane. Hence it follows from (1) that the tangent to the regular curve C at z_0 is rotated through the angle ψ by the transformation $w = f(z)$.



Now, let C_1 and C_2 be two regular curves passing through $z_0 \in D$ and lying in D and C_1' and C_2' be their image curves under the map $w = f(z)$.

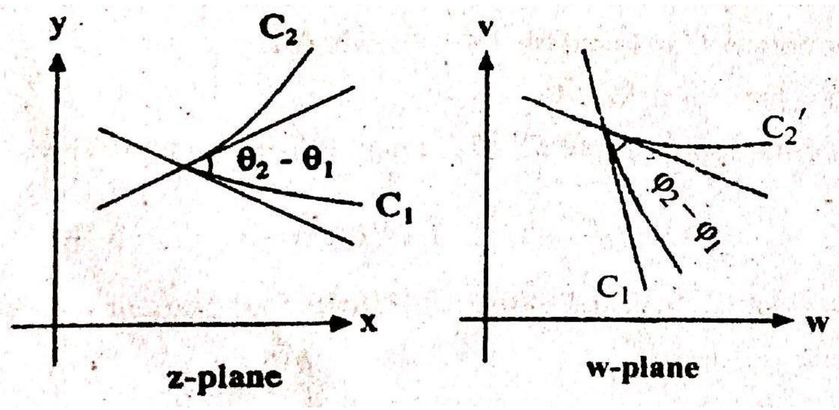
Let θ_1 and θ_2 be the angles made by the tangents to the curves C_1 and C_2 respectively it z_0 with the positive direction of the x-axis in the z-plane.

Let φ_1 and φ_2 be the angles made by the tangents to the cures C_1' and C_2' at $f(z_0)$ in the w-plane.

$\therefore \varphi_1 = \psi + \theta_1$ and $\varphi_2 = \psi + \theta_2$.

$\therefore \varphi_2 - \varphi_1 = \theta_2 - \theta_1$.

$\therefore$ The angle $\varphi_2 - \varphi_1$ from C_1' to C_2' is the same in magnitude and sense as the angle $\theta_2 - \theta_1$ from C_1 to C_2 (refer figure).



Hence the function f preserves angle between the curves C_1 and C_2 at z_0 both in magnitude and direction.

Hence f is conformal at z_0.

Note 1. Under the conformal mapping $w = f(z)$ angle of rotation ψ at z_0 is arg $w'(z_0)$ and the scale factor is $|f'(z_0)|$

Note 2. The condition for conformality at a point z_0, can also be written as $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$. Since $f'(z_0) \neq 0$ we have $|f'(z_0)| \neq 0$.

$\therefore \frac{\partial(u,v)}{\partial(x,y)} \neq 0$ (refer remark 1 in theorem 2.5)

Example 6:

Consider the mapping $w = \bar{z}$. Geometrically it represents reflection about the real axis and its preserves the angle in magnitude but reverses the direction. Hence it is an isogonal mapping.

Example 7:

Consider the mapping $w = z^2$. Angle between any two curves passing through the origin is doubled by this mapping.

Hence the mapping is not conformal at $z = 0$. We notice that $f'(0) = 0$.

Definition. Let $f(z)$ be an analytic function defined in D and let $z_0 \in D. z_0$ is called a critical point of $f(z)$ if $f'(z_0) = 0$.

Solved Problems

Problem 1:

Determine the angle of rotation and scale factor at the point $z = 1 + i$ under the mapping $w = z^2$.

Solution:

We know that the angle of rotation at $1 + i$ under $w = z^2$ is given by arg $w'(1 + i)$ and the scale factor $|w'(1 + i)|$.

Now $w = z^2 \Rightarrow w'(z) = 2z$.

$\therefore w'(1 + i) = 2(1 + i)$.

$\therefore \arg w'(1 + i) = \arg[2(1 + i)] = \tan^{-1}(1) = \pi/4$.

$\therefore$ The angle of rotation at $1 + i$ is $\pi/4$.

Now the scale factor at $z = 1 + i$ is given by

$|w'(1 + i)| = |2(1 + i)| = 2\sqrt{2}$.

Problem 2:

Find the points where the following mappings are conformal. Also find the critical points if any.

(i) $w = z^n$ (n positive integer)

(ii) $w = \frac{1}{z}$

(iii) $w = z + \frac{1}{z}$

(iv) $w = e^z$

(v) $w = \sin z$

(vi) $w = \cosh z$

(vii) $w = az + b$ and $a \neq 0$.

Solution:

We know $w = f(z)$ is conformal at a point z_0 if (a) w is analytic at z_0 and $f'(z_0) \neq 0$.

Also we know z_0 is a critical point of $w = f(z)$ if

(a) w is analytic at z_0 and (b) $f'(z_0) = 0$.

(i) $w = z^n$ (n is a positive integer)

$f(z) = z^n$ is analytic at all points.

Now $f'(z) = nz^{n-1}$ and $f'(z) = 0$ if and only if $z = 0$.

Hence the mapping is conformal at all points $z \neq 0$ and 0 is the only critical point of $f(z)$.

(ii) $w = \frac{1}{z}$. The mapping $f(z) = \frac{1}{z}$ is analytic at all points except $z = 0$.

Now $f'(z) = -\frac{1}{z^2}$ and $f'(z) \neq 0$ for all $z \neq 0$.

Hence the mapping is conformal at all points except $z = 0$ and there is no critical point for the mapping.

(iii) $w = z + \frac{1}{z}$.

$f(z) = z + \frac{1}{z}$ is not analytic at $z = 0$. Hence it is not conformal at $z = 0$.

Now $f'(z) = 1 - \frac{1}{z^2}$.

$f'(z) = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$.

The only critical points are 1 and -1 .

(iv) $w = e^z$.

$f(z) = e^z$ is analytic at all points in the complex plane.

Now $f'(z) = e^z$ and $e^z \neq 0$ for all z^*.

$\therefore$ $f(z)$ is conformal in the entire complex plane and there is no critical point.

(v) $w = \sin z$.

Then $f'(z) = \cos z$. Clearly $f'(z) = \cos z = 0$ if $z = \pi/2 + n\pi$ where $n \in Z$. These are the critical points and $f(z)$ is conformal at all other points.

(vi) $w = \cosh z$.

Hence $f'(z) = \sinh z$. The mapping is not conformal at points where $\sinh z = 0$ and the points are $z = 0, \pm\pi i, \pm 2\pi i, \ldots$ and these points are the critical points.

(vii) $w = az + b (a \neq 0)$

$f(z) = az + b$ is analytic everywhere. Now $f'(z) = a \neq 0$.

$\therefore f'(z) \neq 0$ for all z. Hence this mapping is conformal everywhere and the are no critical points.

Exercises

1. Find the angle of rotation and scale factor for the transformation $w = \frac{1}{z}$ at(i)$z = 1$ and (ii) $z = i$.
2. Show that under the mapping $w = z^2$ the angle of rotation at $z = 2 + i$ is $\tan^{-1}(1/2)$ and the scale factor is $2\sqrt{5}$.
3. Find the angle of rotation and scale factor for the mapping $w = z^3 + 8iz$ at $z = 1 - i$.
4. Find the coefficient of magnification and angle of rotation for (i) $w = z^3$ at $1 + i$ and (ii) $w = (1 - i)z$ at any point z.
5. Show that $w = iz$ represents a rotation through an angle $\pi/2$ and $w = -z$ a rotation through π.
6. Find where the following mappings are conformal and also find the critical points if any.

   (i) $w = z^3$

   (ii) $w = \cos z$

   (iii) $w = \sinh z$

Answers.

1. (i) π - (ii) 0

2. $\tan^{-1}\left(\frac{1}{4}\right); \sqrt{68}$

3. (i) $6; \frac{\pi}{2}$ (ii) $\sqrt{2}; -\frac{\pi}{4}$

4. (i) Conformal at all points except $z = 0$. Origin is a critical point.

   (ii) Conformal except $z = 0, \pm\pi, \pm 2\pi,$ .... These are the critical points.

   (iii) Conformal except at $z = \pm\left(\frac{\pi i}{2}\right), \frac{3\pi i}{2},$ ... These are the critical points.

Bilinear Transformations:

Introduction

A function $f: \boldsymbol{C} \to \boldsymbol{C}$ can be thought of as a transformation from one complex plane to another complex plane. Hence the nature of a complex function can be described by the manner in which it maps regions and curves from one complex plane to another. In this chapter we shall discus's bilinear transformations and see how various regions are transformed by these transformations.

2.2. Elementary Transformations

1.Translation: $w = z + b$

Consider the transformation $w = z + b$.

If $z = x + iy, w = u + iv$ and $b = b_1 + ib_2$ then the image of the point (x, y) in the z-plane is the point $(x + b_1, y + b_2)$ in the w-plane.

Under this transformation the image of any region is simply a translation of that region. Hence the two regions have the same shape, size and orientation. In particular the image of a straight line is a straight line and the image of a circle with center a and radius r is a circle with centre $a + b$ and radius r.

We note that ∞ is the only fixed point of this transformation when $b \neq 0$.

2. Rotation: $w = az$ where $|a| = 1$

Consider the transformation $w = az$ where $|a| = 1$.

Let $z = re^{i\theta}$ and $a = e^{i\alpha}$ so that $|a| = 1$.

$\therefore\ w = az = e^{i\alpha}(re^{i\theta}) = re^{i(\theta+\alpha)}$.

$\therefore$ A point with polar coordinates (r, θ) in the z-plane is mapped to the point $(r, \theta + \alpha)$ in the w-plane. Hence this transformation represents a rotation through an angle $\alpha = \arg a$ about the origin.

Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the two fixed points of this transformation.

3. Magnification or Contraction: $w = bz(b > 0$, real)

Consider the transformation $w = bz$ where b is real and $b > 0$.

Then a point with polar coordinates (r, θ) in the z-plane is mapped into the point (br, θ) in the w-plane. Hence this transformation represents a magnification or contraction by the factor according as $b > 1$ or $b < 1$.

Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the fixed points of this transformation.

Note. In general, the transformation $w = bz$ where b is a non-zero complex number represents a rotation through an angle arg b followed by a magnification or a contraction by the factor $|b|$. Such a transformation is called a homothetic transformation.

4. Inversion: $\boldsymbol{w = \frac{1}{z}}$

Consider the transformation $w = \frac{1}{z}$

Put $z = re^{i\theta}$.

$\therefore\ w = (1/r)e^{-i\theta}$.

This transformation can be expressed as a product of two transformations

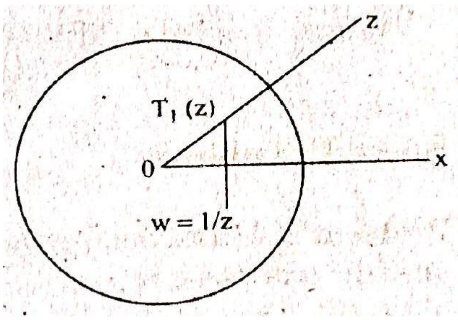
$T_1(z) = (1/r)e^{i\theta}$ and $T_2(z) = re^{-i\theta} = \bar{z}$.

For,
$$\begin{aligned}(T_1 \circ T_2)(z) &= T_1(T_2(z)) \\ &= T_1(re^{-i\theta}) \\ &= \left(\frac{1}{r}\right)e^{-i\theta} = \frac{1}{z}\end{aligned}$$

The transformation $T_1(z) = (1/r)e^{i\theta}$ represents the inversion with respect to the unit circle $|z| = 1$ and $T_2(z) = \bar{z}$ represents reflection about the real axis.

Hence the transformation $w = \frac{1}{z}$ is the inversion w.r.t the unit circle followed by the reflection about the real axis.

Here points outside the unit circle are mapped into



points inside the unit circle and vice versa. Points on the circle are reflected about the real axis.

In terms of cartesian coordinates the above transformation can be expressed in the form $w = u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$.

$$\therefore u = \frac{x}{x^2 + y^2} \text{ and } v = \frac{-y}{x^2 + y^2} .(1)$$

Similarly, from $z = \frac{1}{w}$ we get $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$ .... (1)

Now, consider the equation $a(x^2 + y^2) + bx + cy + d = 0$ .... (2)

where a, b, c, d are real.

This equation represents a circle or a straight line according as $a \neq 0$ or $a = 0$.

Using (1) in (2) we get

$d(u^2 + v^2) + bu - cv + a = 0$ ……….(3)

Now, suppose $a \neq 0; d \neq 0$.

In this case both (2) and (3) represent circles not passing through the origin.

Hence circles not passing through the origin are mapped into circles not passing through the origin.

Similarly, a circle passing through the origin is mapped into a straight line not passing through the origin.

A straight line not passing through the origin is mapped into a circle passing through the origin.

A straight line passing through the origin is again mapped into a line passing through the origin.

Thus, we see that under the transformation $w = \frac{1}{z}$ the image of a circle need not be a circle and the image of a straight line need not be a straight line.

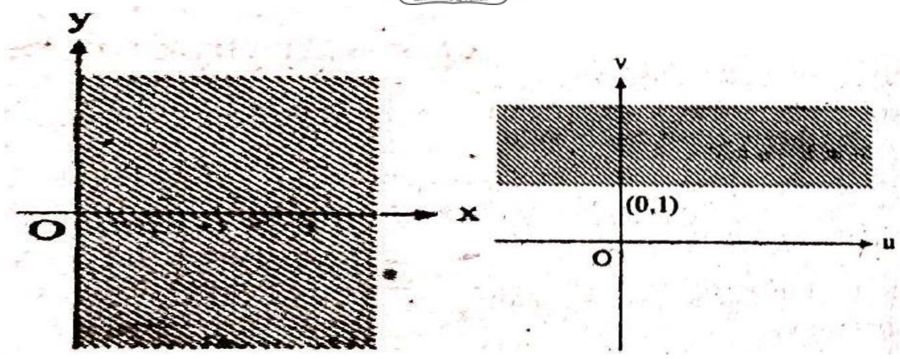
However, the family of circles and lines are again mapped into the family of circles and lines.

We note that the fixed points of the transformation $w = \frac{1}{z}$ are 1 and -1.

Solved Problems

Problem 1:

Under the transformation $w = iz + i$ show that the half plane $x > 0$ maps onto the half plane $v > 1$.



Solution:

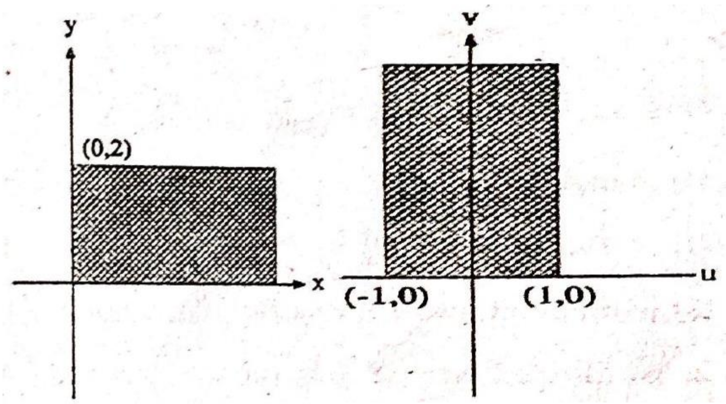
Let $z = x + iy$ and $w = u + iv$.

$w = iz + i \Rightarrow w = i(x + iy) + i = -y + i(x + 1)$.
$\therefore u + iv = -y + i(x + 1)$.
$\therefore u = -y$ and $v = x + 1$.
$\therefore x > 0 \Leftrightarrow v > 1$.

$\therefore$ The half plane $x > 0$ is mapped into the half plane $v > 1$.

Problem 2:

Show that the region in the z-plane given by $x > 0$ and $0 < y < 2$ is mapped into the region in the w-plane given by $-1 < u < 1$ and $v > 0$ under the transformation $w = iz + 1$.



z-plane w-plane

Solution:

Let $z = x + iy$ and $w = u + iv$.

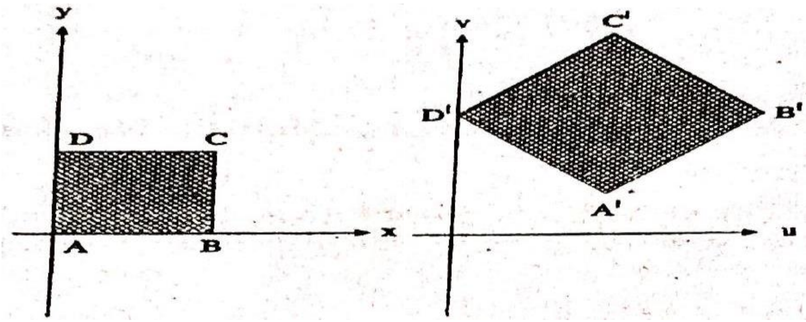
$w = iz + 1 \Rightarrow w = i(x + iy) + 1$
$\Rightarrow u + iv = (-y + 1) + ix.$
$\therefore u = 1 - y$ and $v = x.$
$\therefore x > 0$ and $0 < y < 2 \Leftrightarrow v > 0$ and $-1 < u < 1.$

Hence the given region is mapped into the region $v > 0$ and $-1 < u < 1$ as shown in the figure.

Problem 3:

Find the image of the square region with vertices $(0,0), (2,0), (2,2), (0,2)$ under the transformation

$w = (1 + i)z + (2 + i).$



z-plane **w-plane**

Solution:

$w = (1 + i)z + (2 + i)$. Under this transformation

$A(0,0)$ is mapped into $A' = (1 + i)(0 + 0i) + (2 + i) = 2 + i.$
$B(2,0)$ is mapped into $B' = (1 + i)(2 + 0i) + (2 + i) = 4 + 3i.$
$C(2,2)$ is mapped into $C' = (1 + i)(2 + 2i) + (2 + i) = 2 + 5i.$
$D(0,2)$ is mapped into $D' = (1 + i)(0 + 2i) + (2 + i) = 0 + 3i.$

$\therefore$ The required image region is another square $A'B'C'D'$ as given in the figure.

Note. Since the given transformation is combination of $w_1 = (1 + i)z$ and the translation $w = w_1 + (2 + i)$ we get the image of the square $ABCD$ by increasing each length of $ABCD$ by $\sqrt{2}$ times and rotating the square through an angle 45° about A and then translating it such that A coincides with $A'(2 + i)$.

Problem 4:

Show that by means of the inversion $w = \frac{1}{z}$ the circle given by $|z-3| = 5$ is mapped into the circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$.

Solution. The circle $|z-3| = 5$ is mapped into $\left|\frac{1}{w} - 3\right| = 5$.

Now $\left|\frac{1}{w} - 3\right| = 5 \Rightarrow \left|\frac{1}{u+iv} - 3\right| = 5$

$$\Rightarrow |(1-3u) - 3iv| = 5|u+iv|$$

$$\Rightarrow (1-3u)^2 + 9v^2 = 25(u^2+v^2)$$
$$\Rightarrow 9u^2 - 6u + 1 + 9v^2 = 25u^2 + 25v^2$$
$$\Rightarrow 16(u^2+v^2) + 6u - 1 = 0$$
$$\Rightarrow u^2 + v^2 + \frac{6}{16}u - \frac{1}{16} = 0.$$

This is a circle with center $\left(-\frac{3}{16}, 0\right)$ and radius $\sqrt{\left(\frac{3}{16}\right)^2 + \frac{1}{16}} = \frac{5}{16}$.

Hence the image circle in the w-plane is given by the equation

$\left|w + \frac{3}{16}\right| = \frac{5}{16}$.

Problem 5:

Find the image of the circle $|z - 3i| = 3$ under the map $w = \frac{1}{z}$.

Solution. The image of the circle $|z-3i| = 3$ under the transformation $w = \frac{1}{z}$ is given by the equation $\left|\frac{1}{w} - 3i\right| = 3$.

Now

$$\left|\frac{1}{w} - 3i\right| = 3 \Rightarrow \left|\frac{1}{u+iv} - 3i\right| = 3$$
$$\Rightarrow |1 - 3i(u+iv)| = 3|u+iv|$$
$$\Rightarrow |(1+3v) - i3u| = 3|u+iv|$$
$$\Rightarrow (1+3v)^2 + 9u^2 = 9(u^2+v^2)$$
$$\Rightarrow 1 + 6v + 9v^2 + 9u^2 = 9u^2 + 9v^2$$
$$\Rightarrow 6v + 1 = 0$$

Hence the image of the circle $|z-3i| = 3$ under $w = \frac{1}{z}$ in the z-plane is the straight line $6v + 1 = 0$ in the w-plane.

Problem 6:

Find the image of the strip $2 < x < 3$ under $w = \frac{1}{z}$.

Solution. The transformation $w = \frac{1}{z}$ can be written in Cartesian coordinates as

$$x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}.$$

$$\text{Now, } x > 2 \Rightarrow \frac{u}{u^2 + v^2} > 2$$

$$\Rightarrow 2(u^2 + v^2) - u < 0$$

$$\Rightarrow u^2 + v^2 - \frac{u}{2} < 0.$$

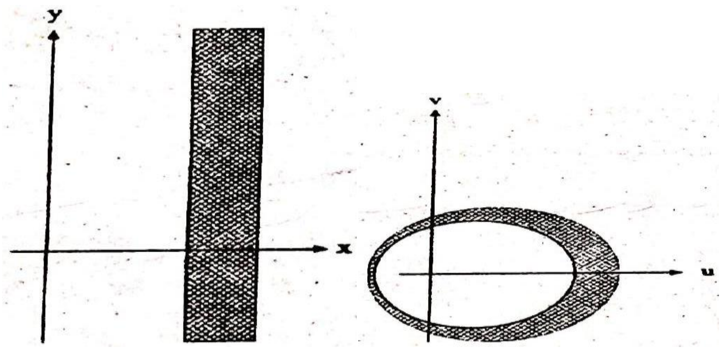
∴ The region $x > 2$ is mapped into a region represented by $u^2 + v^2 - \frac{u}{2} < 0$, which is the interior of the circle with centre $\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{2}$.

Now,

$$x < 3 \Rightarrow \frac{u}{u^2 + v^2} < 3$$

$$\Rightarrow 3(u^2 + v^2) - u > 0$$

$$\Rightarrow u^2 + v^2 - \frac{u}{3} > 0$$



z-plane w-plane

∴ The region $x < 3$ is mapped onto the exterior of the circle with centre $(1/6,0)$ and radius $\frac{1}{\sqrt{6}}$.

∴ The strip $2 < x < 3$ is mapped onto the region bounded by the circles $u^2 + v^2 = \frac{u}{2}$ and $u^2 + v^2 = \frac{u}{3}$ in the w-plane.

Exercises

1.Find the image of the square region with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ under the transformation $w = z + (3 - i)$.

2.Find the image of the rectangular region bounded by $x = 0; y = 0; x = 2; y = 1$ under the (i) translation $w = z + (1 - 2i)$ (ii) rotation $w = iz$ (iii) transformation $w = (1 + i)z + (2 - i)$.

3.Find the image of the strip $0 < x < 1$ under the transformation $u = iz$.

4.Find the image of the region $y > 1$ under the transformation $w = iz + 1$.

5.Find the image of the semi-infinite strip $x > 0; 0 < y < 2$ under the transformation $w = iz + 1$.

6.Describe geometrically the transformation $w = \frac{1}{z-1}$.

7.Describe geometrically the transformation $w = \frac{i}{z}$.

8.Show that by means of the inversion $w = \frac{1}{2}$ the circle given by $|z - 2| = 7$ is mapped into the circle $\left|w + \frac{2}{45}\right| = \frac{7}{45}$.

9.Prove that under the transformation $w = \frac{1}{z}$ if a circle C is transformed into the circle C_1 then the centre of the C need not be transformed into the centre of the circle C_1.

10.Find the image of the following under $w = \frac{1}{z}$.

(i) the circle $|z| = 1$

(ii) the circle $|z - 2i| = 2$

(iii) the strip $1 < x < 2$.

Answers.

2.(i) The rectangular region bounded by $u = 1, v = -2, u = 3$ and $v = -1$

(ii) The rectangular region bounded by $u = 0, v = 0, u = -6$ and $v = 2$

(iii) The rectangular region bounded by $u+v=1, v-u=-3, u+v=5$ and $v-u=-1$

3.The strip $0<v<1$.

4.$u+v>2$

5.$-1<u<1, u>0$.

10.(i) $u=\frac{1}{2}$ (ii) $4v+1=0$ (iii) The region between the circles $u^2+v^2-u=0$ and $u^2+v^2-\frac{u}{2}=0$

2.3 Bilinear Transformations

A transformation of the form $w=T(z)=\frac{az+b}{cz+d}$ where a, b, c, d are complex constants and $ad-bc \neq 0$, is called a bilinear transfor. mation or Mobius transformation.

We define $T(\infty)=\frac{a}{c}$ and $T\left(\frac{-d}{c}\right)=\infty$. Hence T becomes a $1-1$ onto map of the extended complex plane onto itself.

The inverse of (1) is given by

$$z=T^{-1}(w)=\frac{-dw+b}{cw-a}$$

which is also a bilinear transformation.

Note. All the elementary transformations discussed in 3.1 are bilinear transformations.

Theorem 1:

Any bilinear transformation can be expressed as a product of translation, rotation, magnification or contraction and inversion.

Proof:

Let $w=T(z)=\frac{az+b}{cz+d}$ where $ad-bc \neq 0$

be the given bilinear transformation.

case i. $c = 0$.

Hence $d \neq 0$ (since $ad - bc \neq 0$),

$$\therefore \ (1) \Rightarrow w = \frac{az + b}{d}$$
$$= (a/d)z + (b/d).$$

Now, let $T_1(z) = (a/d)z$ and $T_2(z) = z + (b/d)$.

T_1 and T_2 are elementary transformations and

$$(T_2 \circ T_1)(z) = T_2[(a/d)z] = (a/d)z + (b/d)$$
$$= T(z)$$

case ii. $c \neq 0$.

$$w = \frac{az + b}{cz + d} = \frac{a[z + (d/c)] + b - (ad/c)}{c[z + (d/c)]}$$
$$= \frac{a}{c} + \frac{b - (ad/c)}{cz + d}$$

Now, let $T_1(z) = cz + d$

$$T_2(z) = 1/z$$
$$T_3(z) = \left(b - \frac{ad}{c}\right)z$$
$$T_4(z) = z + (a/c)$$

Then $T(z) = (T_4 \circ T_3 \circ T_2 \circ T_1)(z)$ (verify)

Hence the theorem.

Corollary:

Under a bilinear transformation the family of circles and lines are transformed into the family of circles and lines.

Proof:

Under each of the elementary transformations the family of circles and straight lines are transformed into the family of circles and lines. Hence the result follows.

Solved Problems

Problem 1:

Show that the transformation $w = \frac{5-4z}{4z-2}$ maps the unit circle $|z| = 1$ into a circle of radius unity and centre $-\frac{1}{2}$.

Solution:

$w = \frac{5-4z}{4z-2}$.

$\therefore\ 4wz - 2w = 5 - 4z.$

$\therefore\ (4w + 4)z = 5 + 2w.$

$\therefore\ z = \frac{5+2w}{4w+4}.$

Now, $|z| = 1 \Rightarrow z\bar{z} = 1.$

$$\Rightarrow \left(\frac{5 + 2w}{4w + 4}\right)\left(\frac{5 + 2\bar{w}}{4\bar{w} + 4}\right) = 1.$$
$$\Rightarrow 25 + 4w\bar{w} + 10w + 10\bar{w} = 16w\bar{w} + 16 + 16(w + \bar{w})$$
$$\Rightarrow 12w\bar{w} + 6\bar{w} + 6w - 9 = 0.$$
$$\Rightarrow w\bar{w} + \frac{1}{2}\bar{w} + \frac{1}{2}w - \frac{3}{4} = 0.$$

This represents the equation of the circle with center $-\frac{1}{2}$ and radius $\sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$

Hence the result.

Problem 2:

Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $z\bar{z} - 2(z + \bar{z}) = 0$ into a straight line given by $2(w + \bar{w}) + 3 = 0.$

Solution:

$$w = \frac{2z + 3}{z - 4}$$

$\therefore$ w (z – 4) = 2z + 3

$\therefore\ 2(w - 2) = 3 + 4w$

$$\therefore \ z = \frac{3+4w}{w-2}$$

The image of the circk $\bar{z} - 2(z + \bar{z}) = 0$ is

$$\left(\frac{3+4w}{w-2}\right)\left(\frac{3+4\bar{w}}{\bar{w}-2}\right) - 2\left[\frac{3+4w}{w-2} + \frac{3+4\bar{w}}{\bar{w}-2}\right] = 0$$

On simplification we get $2(w + \bar{w}) + 3 = 0$ which is obviously a struight line.

Problem 3:

Show that $w = \frac{z-1}{z+1}$ maps the imaginary axis in the $z - p$lane onto the circle $|w| = 1$. What portion of the $=$ phane corresponds to the interior of the circle $|w| = 1$.

Solution:

$$|w| = 1 \Leftrightarrow \left|\frac{z-1}{z+1}\right| = 1$$

$$\Leftrightarrow |x - 1| =| z + 11$$
$$\Leftrightarrow |x + iy - 1| = |x + iy + 1|$$
$$\Leftrightarrow (x - 1)^2 + y^2 = (x + 1)^2 + y^2$$
$$\Leftrightarrow x = 0$$

Hence the transformation $w = \frac{z-1}{z+1}$ maps the imaginary axis $x = 0$ onto the unit circle $|w| = 1$.

Also since the point $z = 1$ is mapped to $w = 0$ it follows that the half plane $x > 0$ is mapped onto the interior of the circle $|w| = 1$.

Exercises.

1.Express each of the following bilinear transformations as a product of elementary transformations.

(i) $w = \frac{z-1}{z+1}$ (ii) $w = \frac{3z+2i}{iz+6}$ (iii) $w = \frac{1+i-3z}{2-i+iz}$

2.Prove that the set of all bilinear transformations forms a group under composition of mappings.

3.Show that the transformation $w = \frac{i-iz}{1+z}$ maps the unit circle $|z| = 1$ into the real axis of the w-plane.

4.Show that the transformation $w = \frac{iz+2}{4z+i}$ maps the real axis in the z-plane to a circle in the w-plane. Find the centre and radius of the circle.

5.Prove that if a point on a circle is mapped into ∞ under a bilinear transformation, then this circle is transformed into a straight line. Deduce that a pair of intersecting circles can be transformed into a pair of intersecting straight lines and a pair of circles which touch each other can be transformed into a pair of parallel lines.

Answers

1.(i) $T_1(z) = z + 1, T_2(z) = 1/z, T_3(z) = -2Z, T_4(z) = z + 1,$

(ii) $T_1(z) = iz, T_2 = z + 6, T_3(z) = 1/z, T_4(z) = 20iz, T_5(z) = z - zi$

(iii) $T_1(z) = iz, T_2(z) = z + 2 - i, T_3(z) = 1/z, T_4(z) = (-5i + 2)z, T_5(z) = z + 3i.$

2.4. Cross Ratio:

Definition:

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex plane. The cross ratio of these four points denoted by (z_1, z_2, z_3, z_4) is defined by

$$(z_1, z_2, z_3, z_4) = \begin{cases} \dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} & \text{if none of } z_1, z_2, z_3, z_4 \text{ is } \infty \\ \dfrac{z_1 - z_3}{z_1 - z_4} & \text{if } z_2 \text{ is } \infty \\ \dfrac{z_2 - z_4}{z_1 - z_4} & \text{if } z_3 \text{ is } \infty \\ \dfrac{z_1 - z_3}{z_2 - z_3} & \text{if } z_4 \text{ is } \infty \\ \dfrac{z_2 - z_4}{z_2 - z_3} & \text{if } z_1 \text{ is } \infty \end{cases}$$

Theorem 1:

Any bilinear transformation preserves cross ratio.

Proof:

Let $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ be the given bilinear transformation. Let z_1, z_2, z_3, z_4 be four distinct points. Let their images under this transformation be w_1, w_2, w_3, w_4 respectively.

We assume that all the z_i and w_i are different from ∞.

We claim that $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$

We have $w_i = \frac{az_i+b}{cz_i+d}$ $(i = 1,2,3,4)$.

Now, $w_1 - w_3 = \frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}$

$$= \frac{(ad - bc)(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)} = k_1(z_1 - z_3)(say)$$

Similarly, $w_2 - w_4 = k_2(z_2 - z_4)$.

$$\therefore \ (w_1 - w_3)(w_2 - w_4) = k_1k_2(z_1 - z_3)(z_2 - z_4)$$
$$= k(z_1 - z_3)(z_2 - z_4)$$

Similarly, we can prove that

$$(w_1 - w_4)(w_2 - w_3) = k(z_1 - z_4)(z_2 - z_3)$$

$$\therefore \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

The proof is similar if one of the z_i or w_i is ∞.

Note 1. Four distinct points z_1, z_2, z_3, z_4 are collinear or concyclic iff (z_1, z_2, z_3, z_4) is real.

Further any bilinear transformation preserves cross ratio. Hence it follows that the family of circles and straight lines are mapped into the family of circles and straight lines. This gives another proof for the corollary of theorem 1.

Note 2. The bilinear transformation which maps the three points z_1, z_2, z_3 to three points w_1, w_2, w_3 is given by $(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3)$.

Solved Problems

Problem 1:

Find the bilinear transformation which maps the points

$z_1 = 2, z_2 = i, z_3 = -2$, onto $w_1 = 1, w_2 = i, w_3 = -1$ respectively.

Solution:

Let the image of any point z under the required transformation be w. The required bilinear transformation is given by the equation

$$(w, 1, i, -1) = (z, 2, i, -2)$$

$$\therefore \frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-i)(2+2)}{(z+2)(2-i)}.$$

$$\therefore \frac{2(w-i)}{(w+1)(1-i)} = \frac{4(z-i)}{(z+2)(2-i)}.$$

$$\therefore \frac{(w-i)}{w-iw+1-i} = \frac{2(z-i)}{2z-iz+4-2i}.$$

$\therefore iwz + 6w - 3z - 2i = 0$ (verify).

$$\therefore w(iz+6) = 3z + 2i$$

$$\therefore w = \frac{3z+2i}{iz+6}$$

This is the required bilinear transformation.

Problem 2:

Find the bilinear transformation which maps the points $z = -1, 1, \infty$ respectively on $w = -i, -1, i$.

Solution:

Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio we have

$$(z, -1, 1, \infty) = (w, -i, -1, i).$$

(i.e.) $$\frac{z-1}{-1-1} = \frac{(w+1)(-i-i)}{(w-i)(-i+1)}.$$

$$\therefore (z-1)(w - iw - i - 1) = 4iw + 4i.$$

$$\therefore w[z - 1 - i(z-1) - 4i] = 4i + (i+1)(z-1).$$

$$\therefore w = \frac{(i+1)z + 3i - 1}{(1-i)z - 3i - 1}.$$

Problem 3:

Find the bilinear transformation which maps the points $z_1 = 0, \overline{z_2} = -i$, and $z_3 = -1$ into $w_1 = i, w_2 = 1$ and $w_3 = 0$ respectively.

Solution: Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio we have

$$(z, 0, -i, -1) = (w, i, 1, 0).$$

$$\therefore \frac{(z+i)(0+1)}{(z+1)(0+i)} = \frac{(w-1)(i-0)}{(w-0)(i-1)}.$$

$$\therefore (z+i)w(i-1) = -(w-1)(z+1).$$

$$w(iz - i) = z + 1.$$

$\therefore w = -i\left(\frac{z+1}{z-1}\right)$ which is the required bilinear transformation.

Problem 4:

Determine the bilinear transformation which maps $0, 1, \infty$ into $i, -1, -i$ respectively. Under this transformation show that the interior of the unit circle of the z-plane maps onto the half plane left to the v axis (left half of the w-plane).

Solution. The required bilinear transformation is given by the equation

$$(w, i, -1, -i) = (z, 0, 1, \infty).$$

$$\therefore \frac{(w+1)(i+i)}{(w+i)(i+1)} = \frac{z-1}{0-1}.$$

$$\therefore \frac{2i(w+1)}{iw + w - 1 + i} = 1 - z.$$

$$\therefore 2iw + 2i = iw + w - 1 + i - iwz - wz + z - iz.$$

$$\therefore w(2i - i - 1 + iz + z) = z - iz - 2i - 1 + i.$$

$$\therefore w[(i-1) + (i+1)z] = z(1-i) - (1+i).$$

$$\therefore w = \frac{z(1-i) - (1+i)}{z(1+i) - (1-i)}$$

$$\therefore z - \left(\frac{1+i}{1-i}\right)$$

$$= \frac{z - \left(\frac{1-i}{1+i}\right)}{z - i}$$

$$= \frac{z - (1/i)}{z - (1/z - i} = \frac{z - i}{z + i}$$

∴ The required bilinear transformation is $w = \frac{z-i}{z+i}$.

The equations of the left half of the w-plane and the interior of the unit circle in the z-plane are Re$w < 0$ and $|z| < 1$ respectively. Now,

$$\begin{aligned}
\text{Re } w < 0 &\Leftrightarrow \text{Re}\left(\frac{z-i}{z+i}\right) < 0 \\
&\Leftrightarrow \text{Re}\left[\frac{(z-i)(\bar{z}+\bar{\imath})}{|z+i|^2}\right] < 0 \\
&\Leftrightarrow \text{Re}(z-i)(\bar{z}-i) < 0 \\
&\Leftrightarrow \text{Re}(z\bar{z}-i(z+\bar{z})-1) < 0 \\
&\Leftrightarrow \text{Re}(z\bar{z})-1 < 0 (\because i(z+\bar{z}) \text{ is imaginary }) \\
&\Leftrightarrow |z|^2 < 1 \\
&\Leftrightarrow |z| < 1
\end{aligned}$$

∴ The left half plane is mapped into the interior of the unit circle.

Problem 5:

Find the bilinear transformation which maps -1,0,1 of the z-plane onto -1,-i,1 of the w plane. Show that under this transformation the upper half of the z-plane maps onto the interior of the unit circle |m| = 1.

Solution:

The required bilinear transformation is given by the equation

$$(w,-1,-i,1) = (2,-1,0,1).$$

$$\therefore \frac{(w+i)(-1-1)}{(w-1)(-1+i)} = \frac{(z-0)(-1-1)}{(z-1)(-1-0)}$$

$$(\text{ i.e. }) \frac{-2(w+i)}{(w-1)(i-1)} = \frac{-2z}{1-z}$$

$$\therefore (1-z)(w+i) = z(w-1)(i-1)$$

$$(\text{ i.e. }) w+i = iwz+z$$

$$(\text{ i.e. }) w(1-iz) = z-i.$$

$$\therefore w = \frac{z-i}{1-iz} = \frac{i(z-i)}{i(1-iz)} = i\left(\frac{z-i}{z+i}\right)$$

$\therefore$ The required bilinear transformation is $w = i\left(\frac{z-i}{z+i}\right)$.

The equations of the upper half of the w-plane and the interior of the unit circle in the z-plane are given by $\mathrm{Im}w > 0$ and $|z| < 1$ respectively.

Now,

$$\mathrm{Im}w > 0 \Leftrightarrow \mathrm{Im}\left(i\left(\frac{z-i}{z+i}\right)\right) > 0$$

$$\Leftrightarrow \ \mathrm{Re}\left(\frac{z-i}{z+i}\right) < 0$$

$$\Leftrightarrow \ |z| < 1 \text{ (see problem 4).}$$

Hence the upper half plane is mapped into the interior of the unit circle.

Exercises

1.Find the bilinear transformation which maps z_1, z_2, z_3 to w_1, w_2, w_3 respectively where

(a) $z_1 = 2, z_2 = i, z_3 = -2; w_1 = 1, w_2 = i, w_3 = -1$

(b) $z_1 = -i, z_2 = 0, z_3 = i; w_1 = -1, w_2 = i, w_3 = 1$

(c) $z_1 = \infty, z_2 = i, z_3 = 0; w_1 = 0, w_2 = i, w_3 = \infty$

(d) $z_1 = 1, z_2 = -i, z_3 = 0; w_1 = 0, w_2 = 2, w_3 = -i$

(e) $z_1 = 1, z_2 = -1, z_3 = \infty; w_1 = 1 + i, w_2 = 1 - i, w_3 = 1$

(f) $z_1 = 1, z_2 = i, z_3 = -1; w_1 = i, w_2 = 0, w_3 = -i$

(g) $z_1 = -1, z_2 = 1, z_3 = \infty; w_1 = -i, w_2 = -1, w_3 = i$

2.Find the bilinear transformation which maps the points z_1, z_2, z_3 into the points $w_1 = 0, w_2 = 1$ and $w_3 = \infty$.

3.Prove that there exists a unique bilinear transformation mapping three distinct points z_1, z_2, z_3 onto three distinct points w_1, w_2, w_3.

4.Let C_1 be a circle or a straight line in the complex plane. Let C_2 be a circle or a straight line. Prove that there exists a bilinear transformation that maps C_1 onto C_2. (Hint. Choose any three points on C_1 and any three points on C_2).

5.Find a bilinear transformation which maps the vertices $1+i, -i, 2-i$ of a triangle of the z-plane into the points $0,1,i$ of the w-plane.

Answers.

(a) $w = \frac{3z+2i}{iz+6}$ (b) $w = \frac{i(1-z)}{z+1}$ (c) $w = \frac{-1}{z}$

(d) $w = \frac{2(z-1)}{1+iz-2}$ (e) $w = \frac{z+i}{z}$ (f) $w = \frac{i-z}{i+z}$

(g) $w = i\left[\frac{z+(1+2i)}{z+(1-2i)}\right]$

2. $w = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

5. $w = \frac{z-(2+2i)}{(i-1)z-(3+5i)}$

2.5. Fixed Points of Bilinear Transformations

If $w = f(z)$ is any transformation from the z-plane to w-plane, the fixed points of the transformation are the solutions of the equation $z = f(z)$.

Consider a bilinear transformation given by

$$w = \frac{az+b}{cz+d} \text{ where } ad-bc \neq 0.$$

The fixed points or invariant points of the bilinear transformation are given by the roots of the equation $z = \frac{az+b}{cz+d}$.

(i.e.) $cz^2 + (d-a)z - b = 0$.

Case (I) $c \neq 0$.

In this case the fixed points are given by

$$z = \frac{(a-d) \pm \sqrt{[(d-a)^2 + 4bc]}}{2c}.$$

When $(d-a)^2 + 4bc \neq 0$, the given bilinear transformation has two finite fixed points and when $(d-a)^2 + 4bc = 0$ it has only one finite fixed point.

Case (ii) $c = 0$.

In this case the bilinear transformation becomes $w = \left(\frac{a}{d}\right)z + \frac{b}{d}$.

Clearly ∞ is one fixed point.

Other fixed point is determined by the equation $z = \left(\frac{a}{d}\right) z + \frac{b}{d}$.

(i.e.) $(d - a)z - b = 0$.

$\therefore$ If $d - a \neq 0$ we get a finite fixed point $\frac{b}{d-a}$.

If $d - a = 0$ then ∞ is the only fixed point.

Thus we have

Case (i) $c \neq 0$; $(d - a)^2 + 4bc \neq 0 \Rightarrow 2$ finite fixed points.

Case (ii) $c \neq 0$; $(d - a)^2 + 4bc = 0 \Rightarrow$ one finite fixed point.

Case (iii) $c = 0$; $a \neq d \Rightarrow \infty$ and one finite fixed point.

Case (iv) $c = 0; a = d \Rightarrow \infty$ is the only fixed point.

Theorem 1:

Any bilinear transformation having two finite fixed points α and β can be written in the form $\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$.

Proof:

Let T be the given bilinear transformation having α and β as fixed points. Let the image of any point γ under T be δ.

Then the bilinear transformation T is given by $(w, \delta, \alpha, \beta) = (z, \gamma, \alpha, \beta)$.

$$\therefore \frac{(w-\alpha)(\delta-\beta)}{(w-\beta)(\delta-\alpha)} = \frac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}.$$

$$\text{(i.e.) } \frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right) \text{ where } k = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)}. \qquad (1)$$

Definition:

Let T be a bilinear transformation with two finite fixed points α, β. If k given by (1) is real T is called hyperbolic and if $|k| = 1, T$ is called elliptic.

Theorem 3.4. Any bilinear transformation having ∞ and $\alpha \neq \infty$ as fixed points can be written in the form $w - \alpha = k(z - \alpha)$.

Proof:

Let T be the given bilinear transformation having ∞ and α as fixed points. Lef the image of any point γ under T be δ.

Then the bilinear transformation is given by $(w, \delta; \alpha, \infty) = (z, \gamma; \alpha, \infty)$.

$$\therefore \frac{w-\alpha}{\delta-\alpha} = \frac{z-\alpha}{\gamma-\alpha}.$$

$\therefore w - \alpha = k(z-\alpha)$ where $k = \dfrac{\delta-\alpha}{\gamma-\alpha}$.

Definition:

A bilinear transformation with only one finite fixed point is called parabolic.

Theorem 2:

Any bilinear transformation having ∞ as the only fixed point is a translation.

Proof:

Let $w = \frac{az+b}{cz+d}$ be the bilinear transformation having ∞ as the only fixed point.

Then $c = 0$ and $a = d$.

$\therefore$ The bilinear transformation reduces to the form $w = \frac{az+b}{a}$.

$\therefore\ w = z + \left(\frac{b}{a}\right)$ which is a translation.

Theorem 3:

Let C be a circle or a straight line and z_1, z_2 be inverse points or reflection points with respect to C. Let w_1, w_2 and C_1 be the images of z_1, z_2 and C under a bilinear transformation. Then w_1 and w_2 are inverse points or reflection points with respect to C_1.

(i.e) a bilinear transformation preserves inverse points.

Proof:

Let the equation of C be $pz\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0$ ....(1)

Since z_1 and z_2 are inverse points w.r.t. C by theorem 1.11, we have

$$pz_1\bar{z}_2 + \alpha\bar{z}_2 + \bar{\alpha}z_1 + \beta = 0. \dots\dots\dots. (2)$$

Let the given bilinear transformation be $w = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$.

$$\therefore \; z = \frac{dw - b}{-cw + a}.$$

$\therefore$ Under the given bilinear transformation (1) is transformed into

$$p\left(\frac{dw - b}{-cw + a}\right)\left(\frac{\bar{d}\bar{w} - \bar{b}}{-\bar{c}\bar{w} + \bar{a}}\right) + \alpha\left(\frac{\bar{d}\bar{w} - \bar{\beta}}{-\bar{c}\bar{w} + \bar{a}}\right) + \bar{\alpha}\left(\frac{dw - b}{-cw + a}\right) + \beta = 0 \ldots \quad (3)$$

Also (2) is transformed into

$$P\left(\frac{dw_1 - b}{-cw_1 + a}\right)\left(\frac{\bar{d}\bar{w}_2 - \bar{b}}{-\bar{c}\bar{w}_2 + \bar{a}}\right) + \alpha\left(\frac{\bar{d}\bar{w}_2 - \bar{b}}{-\bar{c}\bar{w}_2 + \bar{a}}\right) + \bar{\alpha}\left(\frac{dw_1 - b}{-cw_1 + a}\right) + \beta = 0 \ldots . (4)$$

Clearly (4) is the condition for w_1 and w_2 to be inverse points with respect to (3). Hence the theorem.

Note. We shall regard the center of the circle and ∞ as inverse points with respect to the circle.

Solved Problems

Problem 1:

Find the invariant points of the transformations

(i) $w = \frac{1+z}{1-z}$

(ii) $w = \frac{1}{z-2i}$.

Solution:

(i) The invariant points of $w = f(z)$ are got from $f(z) = z$.

$$\begin{aligned} \therefore f(z) = z \quad \Rightarrow \quad & z = \frac{1+z}{1-z} \\ \Rightarrow \; & z - z^2 = 1 + z \\ \Rightarrow \; & 1 + z^2 = 0 \\ \Rightarrow \; & z = \pm i. \end{aligned}$$

$\therefore$ i and $-i$ are the two fixed points of the transformation.

(ii)
$$\begin{aligned} f(z) = z \quad \Rightarrow \quad & z = \frac{1}{z - 2i} \\ \Rightarrow \quad & z^2 - 2iz - 1 = 0 \\ \Rightarrow \quad & (z - i)^2 = 0 \end{aligned}$$

Hence i is the (only) fixed point.

Problem 2:

Prove that the transformation $w = \bar{z}$ is not a bilinear transformation.

Solution:

Any bilinear transformation, other than the identity transformation has two fixed points. However, the transformation $w = \bar{z}$ has infinitely many fixed points, namely all real numbers. Hence it is not a bilinear transformation.

Note. The above result can also be established by showing that $w = \bar{z}$ does not preserve cross ratio.

Exercises

1. Prove that a bilinear transformation having origin as the fixed point can be written in the form $w = \frac{z}{cz+d}$.
2. Prove that a bilinear transformation having 0 and ∞ as fixed point is of the form $u = az$.
3. Find the fixed points and the normal form of the following bilinear transformations. Also determine whether they are elliptic, hyperbolic or parabolic.

   (i) $w = z + 3$ (ii) $w = 2z + 3$ (iii) $w = \frac{z-1}{z+1}$

   (iv) $w = 6z - 9z$ (v) $w = \frac{z}{2-z}$ (vi) $w = \frac{3z-4}{z-1}$

   (vii) $w = \frac{3iz+1}{z+i}$ (viii) $w = \frac{(2+i)z-2}{z+i}$ (ix) $w = \frac{z-3}{z+1}$.

Answers.

3.(i) ∞ (ii) ∞ (iii) $\pm i; \frac{w-i}{w+i} = \left(\frac{i-1}{i+1}\right)\left(\frac{z-i}{z+i}\right)$; elliptic

(iv) 3; parabolic (v) $0,1; \frac{w}{w-1} = \frac{1}{2}\left(\frac{z}{z-1}\right)$; hyperbolic

(vi) 2; parabolic. (vii) i; parabolic (viii) $1 \pm i$ (ix) $\pm i\sqrt{3}$

UNIT III

Complex Integration: Definite Integral– Cauchy's Theorem–Cauchy integral formula –Higher Derivatives.

Chapter 3: Sections- 3.1 to 3.4

3.Complex Integration

Introduction

In this chapter we develop the theory of integration for complex functions. We assume that the reader is familiar with the Riemann integral of a function defined on $[a, b]$. Using this we define the integeral of a complex valued function defined on $[a, b]$ and the integral of a function $f: D \to \boldsymbol{C}$ where D is a region in $\boldsymbol{C}$, along a curve $\boldsymbol{C}$ lying in D. We prove Cauchy's fundamental theorem and study the various consequences of this theorem.

3.1. Definite integral:

We start with the definition of definite integral for a continuous complex valued function of a real variable.

Definition:

Let $f(t) = u(t) + iv(t)$ be a continuous complex valued function defined on $[a, b]$.

We define $\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$

Remark:

The following properties of the definite integral can be easily verified

1. $\mathrm{Re}\int_a^b f(t)dt = \int_a^b \mathrm{Re}[f(t)]dt$
2. $\mathrm{Im}\int_a^b f(t)dt = \int_a^b \mathrm{Im}[f(t)]dt$
3. $\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt$
4. $\int_a^b cf(t)dt = c\int_a^b f(t)dt$ where c is any complex constant.

Lemma:

$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

Proof:

Let $\int_u^h f(t)dt = re^{i\theta}$.

$$\therefore \int_a^b f(t)dt = r = e^{-i\theta}\int_a^b f(t)dt$$

$$= \text{Re}\left(e^{-i\theta}\int_a^b f(t)dt\right) \text{ (since r is real)}$$

$$= \text{Re}\left(\int_a^b e^{-i\theta} f(t)dt\right) \text{ (using 4)}$$

$$= \int_a^b \text{Re}\left(e^{-i\theta} f(t)dt\right) \text{ (using 1)}$$

$$\leq \int_a^b \left|e^{-i\theta} f(t)\right|dt$$

$$= \int_a^b \left|e^{-i\theta}\right||f(t)|dt$$

$$= \int_a^b |f(t)|dt$$

Thus $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

Definition:

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \leq t \leq b$. Let $f(z)$ be a continuous complex valued function defined in a region containing the curve C. We define

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Example 1: Consider $\int_C f(z)dz$ where $f(z) = \frac{1}{z}$ and C is the circle $|z| = r$ described in the positive sense. The parametric equation of the circle $|z| = r$ is given by $z = re^{it}$ where $0 \leq t \leq 2\pi$ and $z'(t) = ire^{it}$

$$\therefore \int_C f(z)dz = \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt$$
$$= i\int_0^{2\pi} dt$$
$$= 2\pi i.$$

Example 2:

In general, $\int_C \frac{dz}{z-a} = 2\pi i$ where C is the circle with centre a radius r given by the equation $z = a + re^{it}, 0 \leq t \leq 2\pi$. For $\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{it}}{re^{it}} dt = i\int_0^{2\pi} dt = 2\pi i$

Theorem 1:

$\int_{-C} f(z)dz = -\int_C f(z)dz.$

Proof:

Suppose the equation of C is given by $z = z(t)$ where $a \leq t \leq b$. We know that the equation of $-C$ is given by

$z(t) = z(b + a - t)$ where $a \leq t \leq b$.

Now, $\int_{-C} f(z)dz = \int_a^b f(z(b + a - t))z'(b + a - t)(-dt)$.

Put $b + a - t = u$.Then $-dt = du$.

Also $t = a \Rightarrow u = b$ and $t = b \Rightarrow u = a$

$$\therefore \int_{-C} f(z)dz = \int_b^a f(z(u))z'(u)du$$
$$= -\int_a^b f(z(u))z'(u)du$$
$$= -\int_C f(z)dz$$

Remark:

The following results are immediate consequences of the definition.

1. Let α be a complex constant. Then $\int_C \alpha f(z)dz = \alpha\int_C f(z)dz$.
2. $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

Definition:

Let C_1 be a differentiable curve with origin z, a and terminus $z2$. Let C_2 be another differentiable curve with origin z_2 and terminus z_3. Then the curve C which consists of C_1 followed by C_2 is a piecewise differentiable curve with origin $z1$ and terminus 23 . This curve is denoted by $C_1 + C_2$.

Remark. If $C = C_1 + C_2$ then $\int_C f(z)dz = \int_{C_1} f(z) + \int_{C_2} f(z)dz$

In general if $C = C_1 + C_2 + \cdots + C_n$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

Definition:

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \le t \le b$. Then the length l of C is defined by

$$l = \int_a^b |z'(t)|dt.$$

Example:

Consider the circle C with centre a and radius r. The parametric equation of C is given by $z = a + re^{it}$ where $0 \le t \le 2\pi$.

$$\therefore z'(t) = ire^{it}$$

$$\therefore l = \int_0^{2\pi} |z'(t)|dt = \int_0^{2\pi} |ire^{it}|dt = \int_0^{2\pi} rdt$$

$$= 2\pi r$$

Theorem 2:

$\left|\int_C f(z)dz\right| \le Ml$ where $M = \max\{|f(z)|/z \in C\}$ and l is the length of C.

Proof:

Suppose C is given by the equation $z = z(t)$ where $a \le t \le b$.

By definition of M we have

$$|f(z(t))| \le M \text{ for all } t; a \le t \le b \qquad (1)$$

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right|$$

$$\leq \int_a^b |f(z(t))z'(t)|dt$$

$$= \int_a^b |f(z(t)||z'(t))|dt$$

Now,

$$\leq \int_a^b M|z'(t)|dt. \text{(using(1))}$$

$$= M\int_a^b |z'(t)|dt$$

$$= Ml$$

$$\therefore \left|\int_C f(z)dz\right| \leq Ml \leq Ml$$

Solved Problems

Problem 1:

Evaluate $\int_C f(z)dz$ where $f(z) = y - x - i3x^2$ and C is the line segment from $z = 0$ to $z = 1 + i$.

Solution:

The equation of the line segment C joining $z = 0$ and $z = 1 + i$ is given by $y = x$.

$\therefore$ The parametric equation of C can be taken as $x = t$ and $y = t$ where $0 \leq t \leq 1$. Hence $z(t) = x(t) + iy(t) = t + it$ so that $z'(t) = (1 + i)$.

Now, $f(z(t)) = t - t - i3t^2 = -i3t^2$.

$$\int_C f(z)dz = \int_0^1 f(z(t))z'(t)\mathrm{dt}$$

$$= \int_0^1 -i3t^2(1+i)dt$$

$$= -3i(1+i)\left[\frac{t^3}{3}\right]_0^1 = 1 - i$$

Problem 2:

Evaluate the integral $\int_C (x^2 - iy^2)dz$ where C is the parabola $y = 2x^2$ from (1,2) to (2,8).

Solution:

Let f(z)=$(x^2 - iy^2)$. The parametric equation of C is given by $x = t$ and $y = 2t^2$ where $1 \leq t \leq 2$.

$$\therefore z(t) = x(t) + iy(t) = t + i2t^2 \text{ and } z'(t) = 1 + 4it$$

$$\therefore \int_C (x^2 - iy^2)dz = \int_1^2 (t^2 - 4it^4)(1 + 4it)dt$$

$$= \int_1^2 [(t^2 + 16t^5) + i(4t^3 - 4t^4)]dt$$

$$= \left[\left(\frac{t^3}{3} + \frac{16t^6}{6}\right) + i\left(t^4 - \frac{4t^5}{5}\right)\right]_1^2$$

$$= \frac{511}{3} - \frac{49}{5}i$$

Problem 3:

Evaluate $\int_C \frac{z+2}{z} dz$ where C is the semi circle $z = 2e^{i\theta}$ where $0 \leq \theta \leq \pi$

Solution:

Here $z'(\theta) = 2ie^{i\theta}$ so that $dz = 2ie^{i\theta}d\theta$.

$$\int_C \frac{z+2}{z} dz = \int_0^\pi \left(\frac{2e^{i\theta} + 2}{2e^{i\theta}}\right)\left(2ie^{i\theta}d\theta\right)$$

$$= 2i \int_0^\pi \left(1 + e^{i\theta}\right)d\theta$$

$$= 2i \left[\theta + \frac{e^{i\theta}}{i}\right]_0^\pi$$

$$= 2i \left[\left(\pi - \frac{1}{i}\right) - \left(\frac{1}{i}\right)\right]$$

$$= 2i \left[\frac{\pi i - 2}{i}\right]$$

$$= -4 + 2\pi i$$

Problem 4:

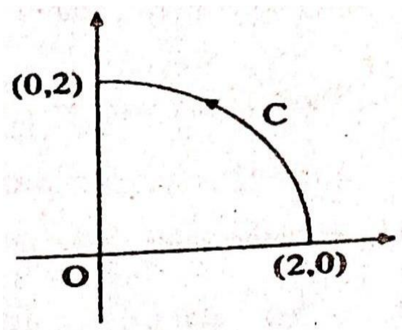
Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without actually evaluting the integral show that

$\left|\int_C \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}$.

Solution:

Let $f(z) = \frac{1}{z^2+1}$.

Since C is the circular arc of radius 2 lying in th - first quardrant the length l of C is given by $l = \frac{1}{4}(2\pi \times 2) = \pi$



Also, on $C, |z^2 + 1| = |z^2 - (-1)| \geq |z^2| - |-1| = |z|^2 - 1 = 3$

Thus $|z^2 + 1| \geq 3$

$\therefore \left|\frac{1}{z^2 + 1}\right| \leq \frac{1}{3}$.

Hence by theorem 2 $\left|\int_C \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}$.

Problem 5:

Without evaluating the integral show that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$ where C is the line segment from $z = i$ to $z = 1$.

Solution:

C is the line segment joining $(0,1)$ to $(1,0)$ and its length is obviously $\sqrt{2}$. As z varies on C, the minimum value of $|z|$ is the perpendicular distance from the origin to the line segment C.

Thus on $C, |z| \geq \frac{1}{\sqrt{2}}$ so that $|z|^4 \geq \frac{1}{4}$.

$\therefore \left|\frac{1}{z^4}\right| \leq 4.$

$\therefore$ By theorem 6.2$\left| \int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}.$

Exercises

1.Evaluate $\int_C xdz$ where C is the directed line segment from 0 to $1+i$

2.Evaluate $\int_C xdz$ where C is the dircle $|z| = r$.

3.Show that $\int_C xdz = \frac{i\pi}{2}$ and $\int_C ydz = -\frac{\pi}{2}$ where C is the semi circle $|z| = 1$ and $0 \leq \arg z \leq \pi$ with inital point $z = 1$.

4.Find the value of the integral $\int_0^{1+i} (x - y + ix^2)dz$

(i) along the straight line from $z = 0$ to $z = 1 + i$.

(ii) along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to the real axis from $z = i$ to $z = 1 + i$.

5.Show that $\int_{C_1} \bar{z}dz = \pi i$ and $\int_{C_2} \bar{z}dz = -\pi i$ if C_1 in the upper half of the circle $|z| = 1$ with $z = 1$ as the initial point and C_2 is the same semicircle with $z = -1$ as the inital point.

6.Evaluate the integrals $I_1 = \int_C xdz$ and $I_2 = \int_C ydz$ along the following paths.

(i) along the radius vector of the point $z = 2 + i$.

(ii) along the semicircle $|z| = 1, 0 \leq \arg z \leq \pi$ with starting point 1 .

(iii) around the circle $|z - a| = r$.

7.Evaluate $\int_C |z|dz$ along the following paths.

(i) around the semicircle $|z| = 1, 0 \leq z \leq \pi$ with starting point 1 .

(ii) around the semicircle $|z| = 1, -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$ with starting point $-i$

(iii) around the circle $|z| = r$.

8.Evlauate $\int_C |z|\bar{z}dz$ where C is the closed curve consisting of the upper semicircle $|z| = 1$ and the segment $-1 \leq x \leq 1, y = 0$.

9.Compute $\int_{|z|=1} |z-1||dz|$

10.Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)$ dy along

(i) the parabola $x = 2t, y = t^2 + 3$

(ii) the straight-line segments from (0,3) to (2,3) and

then from (2,3) to (2,4).

(iii) a straight line from (0,3) to (2,4).

11.Evaluate $\int_C (x+2y)dx + (y-2x)dy$ where C is the ellipse defined by $x = 4\cos\theta, y = 3\sin\theta$ and C is described in the anticlockwise direction.

12.Evlauate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$ along

(i) the curve $y = x^2 + 1$

(ii) the straight line joining (0,1) and (2,5)

(iii) the straight lines from (0,1) to (0,5) and then from (0,5) to (2,5)

(iv) the straight lines from (0,1) to (2,1) and then from (2,1) to (2,5)

13.Compute $\int_C \frac{dz}{z}$ where C denotes

(i) the square described in the positive sense with sides parallel to the axes and of length $2a$ and having its centre at the origin.

(ii) the circle $|z| = r$ described in the positive sense.

14.Evaluate $\int_C |z|dz$ where C is the circle $|z-1| = 1$ in the positive sense.

15.If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1,1) and (2,3) evaluate $\int_C (12z^2 - 4iz)dz$.

Answers

I. $(1+i)/2$ 2. $i\pi r^2$ 4. (i) $(-1+i)/3$ (ii) $-(3+i)/6$

6. (i) $2+i; (2+i)/2$ (ii) $i\pi/2; -\pi/2$ (iii) $i\pi r^2; -\pi r^2$

7. (i) -2 (ii) 2i (iii) 0

8. πi 9. 8 10. (i) $\frac{33}{2}$ (ii) $\frac{103}{6}$ (iii) $\frac{97}{6}$ 11. -48π

12. (i) $\frac{88}{3}$ (ii) 32 (iii) 40 (iv) 24

13. (i) $2\pi i$ (ii) $2\pi i$ 14. $8i/3$ 15. $-156+38i$

3.2. Cauchy's Theorem:

In this section we prove the fundamental theorem of integration known as Cauchy's theorem which forms the basis for the theory of complex integration

Definition:

Let $p(x,y)$ and $q(x,y)$ be two real valued functions. Then the differential equation $p(x,y)dx+q(x,y)dy=0$ is said to be exact if there exists a function $u(x,y)$ such that $\frac{\partial u}{\partial x}=p$ and $\frac{\partial u}{\partial y}=q$.

We assume the following theorem without proof.

Theorem 1:

$\int_C pdx+qdy$ depends only on the end points of C if and only if the integrand is exact.

Remark. The above theorem is true if p and q are complex valued functions as well.

We now apply the above theorem for complex functions to get a charecterisation for $\int_C f(z)dz$ to depend only on end points of C.

Theorem 2:

Let $f(z)$ be a continuous complex valued function defined on a region D. Then $\int_C f(z)dz$ depends only on the end points of C if and only if there exists an analytic function $F(z)$ such that $F'(z)=f(z)$ in D.

Proof:

$\int_C f(z)dz=\int_C f(z)(dx+idy)$(since $z=x+iy$)

$$=\int_C f(z)dx+if(z)dy$$

$\int_C f(z)dz$ depends only on the end points of C if and only if there exists a function $F(z)$ defined on D such that $\frac{\partial F}{\partial x} = f(z)$ and $\frac{\partial F}{\partial y} = if(z)$.

$\therefore \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$ so that $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$ which is the complex form of the Cauchy Riemann equation for $F(z)$

Since $f(z)$ is continuous the partial derivatives of $F(z)$ are also continuous and hence $F(z)$ is analytic in D and $F'(z) = f(z)$. Hence the theorem.

Corollary 1. Let $f(z)$ be a continuous complex valued function defined on a region D then $\int_C f(z)dz = 0$ for every closed curve C lying in D iff there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Corollary 2. $\int_C (z-a)^n dz = 0$ for every closed curve C provided $n \geq 0$.

Proof. Let $F(z) = \frac{(z-a)^{n+1}}{n+1}$.

Clearly $F'(z) = (z-a)^n = f(z)$.

$\therefore$ By corollary (1), $\int_C f(z)dz = 0$. Hence $\int_C (z-a)^n dz = 0$ for all $n \geq 0$.

Lemma 1:

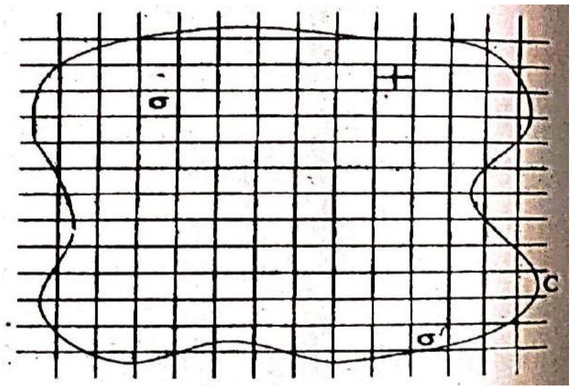
Let C be a simple closed curve. Let D denote the closed region consisting of all points interior to C together with the points on C. Let f be a function analytic in D, Then given $\varepsilon > 0$ it is possible to cover D with a finite number of squares and partial squares whose boundaries are denoted by C_j such that their exists' points z_j lying inside or on each C_j satisfying

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \varepsilon (j = 1,2, \ldots\ldots, n) \qquad (1)$$

for all points z distinct from each z_j and lying inside or on C_j.

Proof:

We subdivide the region D into squares and partial squares by drawing equally spaced lines parallel to the coordinate axes (refer figure). (A square is a closed region consisting of all points on and interior to it. If a particular square contains points which are not in D we remove those points and call what remains a partial square. In this figure σ is a square and σ' is a partial square). This gives a finite number of squares and partial squares which cover the region$D$.



Suppose the Lemma is false. Then in the covering constructed as above there exists a subregion with boundary C_j such that no point z_j exists satisfying (1).

Let σ_0 denote that subregion if it is a square. If it is a partial square let σ_0 denote the entire square of which it is a part.

We now subdivide σ_0 into four smaller squares by drawing line segments joining the mid points of the opposite sides. At least one of the four smaller squares say σ_1 is such that σ_1 contains points of D and no point z_j satisfying (1) exists.

Continuing this process we obtain a nested infinite sequence of squares $\sigma_1, \sigma_2, \sigma_n$ such that for each σ_n no z_j satisfying (1) exists.

Now there exists a point z_0 common to each σ_n such that for any $\delta > 0$ the neight bourhood $|z - z_0| < \delta$ contains all the squares σ_n for all sufficiently large values of n.

Hence every neighborhood of z_0 contains points of D distinct from z_0. Hence z_0 is a limit point of D. Since D is closed $z_0 \in D$.

Since $f(z)$ is analytic at z_0 there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon \dots \dots (2)$$

Choose N such that the square σ_N is contained in the neighbourhood $|z - z_0| < \delta$. Then for every point z in σ_N (2) holds.

$\therefore z_0$ serves as the point z_j stated in the lemma. This is a contradiction since there is no z_j in σ_N satisfying (1).

This contradiction proves the lemma.

Theorem 3: (Cauchy's Theorem)

Let f be a function which is analytic at all points inside and on a simple closed curve C. Then $\int_C f(z)dz = 0$.

Proof:

Let D be the closed region consisting of all points interior to C together with the points on C.

Let $\varepsilon > 0$ be given.

Let $C_j (j = 1,2, \dots \dots, n)$ denote the boundaries of the squares and partial squares covering D such that there exists a point z_j lying inside or on C_j satisfying

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \varepsilon \dots \dots \dots \dots (1)$$

for all z distinct from z_j and lying within or on C_j.

$$\text{Let } \delta_j(z) = \begin{cases} \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$$

Clearly $\delta_j(z)$ is a continuous function and

$$f(z) = f(z_j) - z_j f'(z_j) + z f'(z_j) + (z - z_j)\delta_j(z)$$

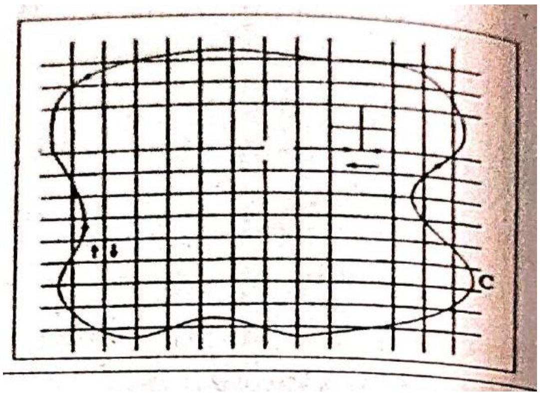
$$\therefore \int_{C_j} f(z)dz = \int_{C_j} f(z_j)dz - \int_{C_j} z_j f'(z_j)dz + \int_{C_j} z f'(z_j)dz + \int_{C_j} (z - z_j)\delta_j(z)dz$$

$$= f(z_j)\int_{C_j} dz - z_j f'(z_j)\int_{C_j} dz + f'(z_j)\int_{C_j} zdz + \int_{C_j} (z - z_j)\delta_j(z)dz$$

$$= \int_{C_j} (z - z_j)\delta_j(z)dz \left(\text{since } \int_{C_j} dz = 0 \text{ and } \int_{C_j} zdz = 0\right)$$

$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz$ ....(2)

Now, in the sum $\sum_{j=1}^{n} \int_{C_j} f(z)dz$ the integrals along the common boundary of every pair of adjacent subregions cancel each other. (since the integral is taken in one direction along that line segment in one subregion and in the opposite direction in the other) (refer figure)



Hence only the integrals along the arcs which are the parts of C remain.

$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \int_{C} f(z)dz.$$

$\therefore$ From (2) $\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz.$

$$\therefore \left|\int_C f(z)dz\right| = \left|\sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz\right|$$

$$\leq \sum_{j=1}^{n} \int_{C_j} |(z - z_j)\delta_j(z)|dz$$

$$= \sum_{j=1}^{n} \int_{C_j} |z - z_j||\delta_j(z)|dz$$

$\therefore \left|\int_C f(z)dz\right| \leq \sum_{j=1}^{n} \int_{C_j} |z - z_j||\delta_j(z)|dz$ ……….(3)

Now if C_j is a square and s_j is the length of its side then $|z - z_j| < \sqrt{2}s_j$ for all z on C_j. Also, from (1) we have $|\delta_j(z)| < \varepsilon$ and hence

$$\int_{C_j} |z - z_j||\delta_j(z)|dz < (\sqrt{2}s_j\varepsilon)(4s_j) \quad \text{(by theorem 2)}$$

$$= 4\sqrt{2}A_j\varepsilon \quad \ldots\ldots\ldots(4)$$

where A_j is the area of the square C_j

Similarly for a partial square with boundary C_j if l_j is the length of the arc of C which forms a part of C_j. We have

$$\int_{C_j} |z - z_j||\delta_j(z)|dz < \sqrt{2}s_j\varepsilon(4s_j + l_j)$$

$$< 4\sqrt{2}A_j\varepsilon + \sqrt{2}Sl_j \ldots\ldots\ldots\ldots(5)$$

where S is the length of a side of some square containing the entire region D as well as all the squares orignially used in covering D.

We observe that the sum of all A_j 's that occur in the right-hand side of (4) and (5) do not exceed S^2 and the sum of all the l_j^s is equal to L (the length of C).

Using (4) and (5) in (3) we obtain

$$\int_C f(z)dz \mid < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon$$

$= k\varepsilon$ where $k = 4\sqrt{2}S^2 + \sqrt{2}SL$ is a constant.

Thus $\left|\int_C f(z)dz\right| < k\varepsilon$.

Since ε is arbitary we have $\int_C f(z)dz = 0$.

Note.

Cauchy's theorem was first proved by using Green's theorem with the additional hypothesis that $f'(z)$ is continuous. Later Goursat proved the theorem without the hypothesis that $f'(z)$ is continuous. For this reason the theorem is sometimes known as Cauchy-Goursat theorem.

Definition:

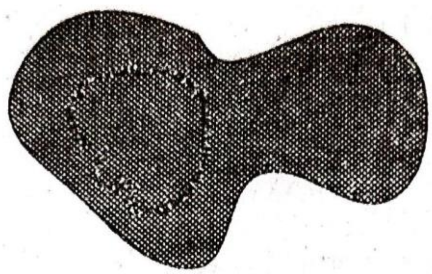
A region D is said to be simply connected if every simple closed curve lying in D encloses only points of D.

For example the interior of a simple closed curve is a simply connected region.

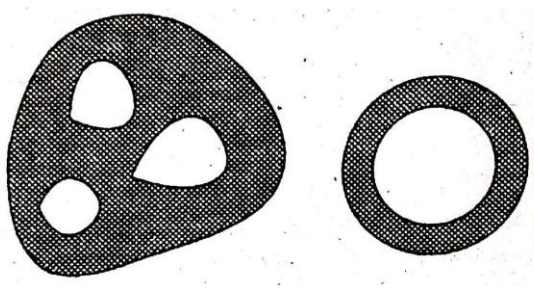
The annular region enclosed by two concentric circles is not simply connected.

A region which is not a simply connected is said to be a multiply connected region.

Intuitively a simply connected region is one which does not have any holes in it.



Simply connected region



Multiply connected regions

We observe that Cauchy's theorem can be restated as follows

Theorem 4: (Cauchy's theorem for simply connected regions).

Let f be a function which is analytic in a simply connected region D. Let C be any simple closed curve lying within D. Then $\int_C f(z)dz = 0$.

We now extend Cauchy's theorem to certain types of multiply connected regions.

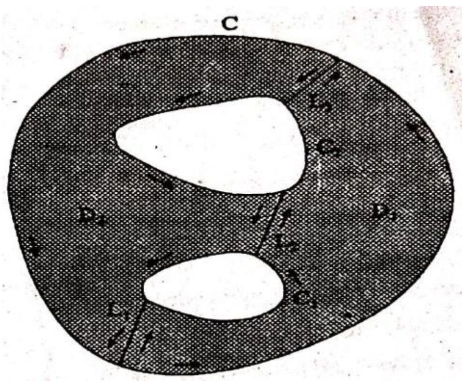
Theorem 5: (Cauchy's theorem for multiply connected regions)

Let C be a simple closed curve. Let $C_j(j = 1,2, \dots \dots, n)$ be a finite number of simple closed curves lying in the interior of C such that the interiors of C_j 's are disjoint. Let D be the closed region consisting of all points within and on C except the points interior to each C_j. Let B denote the entire oriented boundary of D consisting of C and all the C_j described in a direction such that the points of D are to the left of B. Let f be a function which is analytic in D. Then $\int_B f(z)dz = 0$.

Proof:

Let L_1 be a polygonal path joining a point of C to a point of C_1; L_2 a polygonal path joining a point of C_1 to a point of C_2; ... ... ; L_i a polygonal path joining a point of C_{i-1} to a point of C_i and L_{n+1} a polygonal path joining a point of C_n to a point of C such that no two L_i^s cross each other (refer figure).

This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the boundaries of D_1 and D_2 respectively.



By Cauchy's theorem for simply connected region

$$\int_{B_1} f(z)dz = 0 \text{ and } \int_{B_2} f(z)dz = 0.$$

Also $\int_{B_1} f(z)dz \mp \int_{B_2} f(z)dz = \int_B f(z)dz$ since the integrals aiong L_j are taken twice i the opposite directions and cancel each other.

$$\therefore \int_B f(z)dz = 0.$$

We observe that $B = C - C_1 - C_2 - \cdots - C_n$ and hence the above theorem can also be written in the form

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

Note:

In particular if C is a simple closed curve and C_0 is another simple closed curve lying in the interior of C and f is analytic in the region D consisting of all points inside and on C excluding the points interior to C_0 then $\int_C f(z)dz = \int_{C_0} f(z)dz$.

3.3. Cauchy's Integral Formula

In this section we establish another fundamental result known as Cauchy's integral formula using Cauchy's theorem.

Theorem 1:

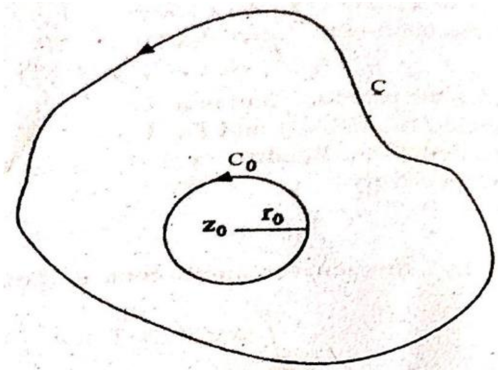
Let $f(z)$ be a function which is analytic inside and on a simple clored curve C. Let z_0 be any point in the interior of C.

Then $f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{z - z_0} dz.$

Proof:

Choose a circle C_0 with centre z_0 and radius r_0 such that C_0 lies in the interior of C.

Now, z_0 is the only point inside C at which the function $\frac{f(z)}{z-z_0}$ is not analytic and hence is analytic in the region D consisting of all points inside and on C except the points interior to C_0.



$$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \frac{f(z)dz}{z-z_0}$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\right) dz$$

Hence $$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz + \int_{C_0} \frac{f(z_0)}{z-z_0} dz$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz + f(z_0)\int_{C_0} \frac{dz}{z-z_0}$$

$$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz + f(z_0)(2\pi i)$$

Thus $$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz + 2\pi i f z_0 \ldots\ldots\ldots (1)$$

We now claim that $\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz = 0.$

Since $f(z)$ is analytic inside and on C it is continuous at $z0 -$

$\therefore$ Given $\varepsilon \sim 0$ there exists $S > 0$ such that

$$|z - z0| < \delta \Rightarrow |f(z) - f(z0)| \leqslant t$$

If we choose $r_0 < \delta$, then $|z - z0| < m \Rightarrow |f(z) - f(z0)| < E,$

Hence $\left|\int_{C_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz\right| < \left(\frac{\varepsilon}{r_0}\right)(2\pi r_0)$(sec 3.2 by theorem 2)

$$= 2\pi\varepsilon$$

Thus $\left|\int_{C_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz\right| < 2\pi s.$

Since ε is arbitrary we have $\int_{C_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)dz = 0.$

$\therefore$ From (1) we get $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0).$

$\therefore f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$

Theorem 2:

Let $f(z)$ be analytic in a region D bounded by two concentric circles C; and C_2 and on the boundary. Let z_0 be any point in D. Then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0} - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}dz.$$

Proof:

Let L_1 and L_2 be two disjoint line segments not passing through z_0 both joining a point of C_1 to a point of C_2 as shown in the figure. This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the oriented boundary of D_1 and D_2 respectively.

Then $B_1 + B_2 = C_1 - C_2$ .... (1)

We assume without loss of generality that $z_0 \in D_1$.

By Cauchy's integral formula,

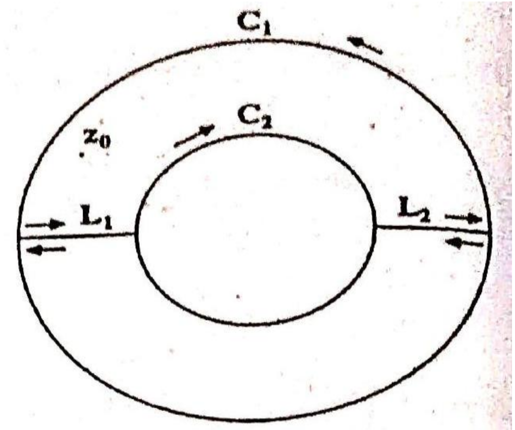
$$\frac{1}{2\pi i}\int_{B_1}\frac{f(z)}{z-z_0}dz = f(z_0) \ldots\ldots(2)$$

Also $\frac{f(z)}{z-z_0}$ is analytic in D_2 and hence by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{B_2}\frac{f(z)}{z-z0}dz = 0 \ldots\ldots(3)$$

Adding (2) and (3) and using (1) we get

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1 - C_2} \frac{f(z)}{z - z_0} dz$$
$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0} dz$$



Example 1:

Consider $\int_C \frac{dz}{z-3}$ where C is the circle $|z - 2| = 5$.

Let $f(z) = 1$

The point $z = 3$ lies inside C.

Hence by Cauchy's integral formula $\int_C \frac{dz}{z-3} = 2\pi i f(3) = 2\pi i$.

Example 2:

Let C denote the unit circle $|z| = 1$

Then $\int_C \frac{e^z}{z} dz = \int_C \frac{e^z}{z-0} dz = 2\pi i e^0 = 2\pi i$.

Theorem 3:

Let $f(z)$ be analytic inside and on the circle C with centre a and radius r. Then $f(a) = \frac{\int_0^l f(z)ds}{l}$ where s is the arc length and l is the circumference of the circle.

(i,e) The value of the function at the center is equal to the mean of the value of the function on the circumference.

Proof:

By Cauchy's integral's formula we have $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)} dz.$

Now the equation of the circle C is given by $z = a + re^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

$$\therefore dz = irc^{i\theta} d\theta$$

$$\therefore f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a + rc^{i\theta})}{re^{i\theta}} (ire^{i\theta} d\theta)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta}) d\theta$$

Also, we have $s = r\theta$ and s varies from 0 to l.

$$\therefore d\theta = \frac{ds}{r}$$

$$\therefore f(a) = \frac{1}{2\pi r}\int_0^{l} f(a + re^{i\theta}) ds$$

$$= \frac{1}{l}\int_0^{l} f(z) ds$$

Hence the theorem.

Theorem 4: (Maximum Modulus Theorem)

Let $f(z)$ be continuous in a closed and bounded region D and analytic and nonconstant in the interior of D. Then $|f(z)|$ attains its maximum value on the boundary of D and never in the interior of D.

Proof. Since f is continuous in a closed and bounded region $D, |f(z)|$ is bounded and attains its bound.

$\therefore$ There exists a positive real number M such that

$$|f(z)| \leq M \text{ for all } z \in D \ldots \ldots \ldots (1)$$

and equality holds for at least one point z in D. Suppose that there exists an interior point $z_0 \in D$ such that

$$|f(z_0)| = M \ldots \ldots \ldots (2)$$

Choose a circle with center z_0 and radius r such that the circular disc $|z - z_0| \leq r$ is contained in D. Then we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)e^{i\theta}d\theta. \text{(refer previous theorem)}$$

$$\therefore |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|d\theta \quad \ldots\ldots\ldots(3)$$

Also, from (1) and (2) we have $\left|f\left(z_0 + re^{i\theta}\right)\right| \leq |f(z_0)|$

$$\therefore \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|d\theta \leq 2\pi|f(z_0)|$$

$$\therefore |f(z_0)| \geq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|d\theta \ldots\ldots\ldots\ldots(4)$$

From (3) and (4) we get $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|d\theta$

$$\therefore\ 2\pi|f(z_0)| = \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|d\theta$$

$$\int_0^{2\pi} |f(z_0)|d\theta = \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|d\theta$$

$$\therefore \int_0^{2\pi} \left[|f(z_0)| - \left|f\left(z_0 + re^{i\theta}\right)\right|\right]d\theta = 0$$

Since the integrand in the above expression is continuous and non- negative we have $|f(z_0)| - \left|f\left(z_0 + re^{i\theta}\right)\right| = 0$

(ie) $|f(z_0)| = \left|f\left(z_0 + re^{i\theta}\right)\right|$ for all z in the circular disc $|z - z_0| < r$.

(ie) $|f(z_0)| = |f(z)|$ for all z in the circular disc.

$\therefore\ f(z)$ is constant in a neighbourhood of z_0.

Since $f(z)$ is continuous it follows that $f(z)$ is constant throughout D which is a contradiction.

$\therefore$ The maximum of $|f(z)|$ is not attained at any of the interior points of D.

Hence the theorem.

Solved Problems

Problem 1:

Evaluate using Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}dz$ where C is $|z| = 4$

Solution:

$f(z) = z^2 + 5$ is analytic inside and on $|z| = 4$ and $z = 3$ lies inside it.

$\therefore$ By Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}dz = f(3) = 3^2 + 5 = 14.$

Problem 2:

Evaluate $\int_C \frac{zdz}{z^2-1}$ where C is the positively oriented circle $|z| = 2.$

Solution:

$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$

$$\therefore \int_C \frac{z}{z^2-1}dz = \frac{1}{2}\int_C \frac{z}{z-1}dz - \frac{1}{2}\int_C \frac{zdz}{z+1}.$$

$f(z) = z$ is analytic and $1, -1$ lie in the interior of C.

$\therefore$ By Cauchy's integral formula $\int_C \frac{zdz}{z-1} = 2\pi i f(1) = 2\pi i.$

$$\text{Also } \int_C \frac{zdz}{z+1} = 2\pi i f(-1) = -2\pi i$$

$$\therefore \int_C \frac{zdz}{z^2-1} = \frac{1}{2}(2\pi i) - \frac{1}{2}(-2\pi i) = 2\pi i$$

Problem 3:

Evaluate $\int_C \frac{e^z}{z^2+4}dz$ where C is positively oriented circle $|z - i| = 2.$

Solution:

$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) \text{ (by partial fraction).}$$

Now, $2i$ lies inside C and by Cauchy's integral formula we have $\int_C \frac{e^z}{z-2i}dz =$ $2\pi i e^{2i}.$

Also $-2i$ lies outside C and hence $\frac{e^z}{z+2i}$ is analytic inside and on C.

Hence by Cauchy's theorem $\int_C \frac{e^z}{z+2i} dz = 0$.

$$\therefore \int_C \frac{e^z}{z^2+4} dz = \frac{1}{4i}\left(2\pi i e^{2i} - 0\right) = \frac{\pi}{2} e^{2i}.$$

Problem 4:

Evaluate $\int_C \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\right) dz$ where C is the circle $|z| = 3$.

Solution:

By partial fractions $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$.

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$. Then $f(z)$ is analytic inside and on C and the points l and 2 lie inside C. Hence by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-1} dz = 2\pi i f(1)$$
$$= 2\pi i(\sin \pi + \cos \pi)$$
$$= -2\pi i$$

Similarly $$\int_C \frac{f(z)}{z-2} dz = 2\pi i f(2)$$
$$= 2\pi i(\cos 4\pi + \sin 4\pi)$$
$$= 2\pi i$$

Hence $\int_C \frac{f(z)}{(z-1)(z-2)} dz = 2\pi i - (-2\pi i) = 4\pi i$

Problem 5:

Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$ where C is described in the positive sense.

Evaluate (i) $\int_C \frac{zdz}{2z+1}$ and (ii) $\int_C \frac{\cos z}{z(z^2+8)}$.

Solution:

(i) $\int_C \frac{zdz}{2z+1} = \frac{1}{2} \int_C \frac{zdz}{z+\frac{1}{2}}$

$$= \frac{1}{2}(2\pi i)\left(-\frac{1}{2}\right) \text{ (by Cauchy's integral formula)}$$

$$= \frac{-\pi i}{2}$$

(ii) Let $f(z) = \frac{\cos z}{z^2+8}$. The points where $f(z)$ is not analytic are $\pm i2\sqrt{2}$ and these points lie outside C. Hence $f(z)$ is analytic inside and on C.

By Cauchy's integral formula

$$\int_C \frac{\cos z}{z(z^2+8)} dz = \int_C \frac{f(z)}{z} dz = 2\pi i f'(0) = 2\pi i \left(\frac{1}{8}\right) = \frac{\pi i}{4}$$

Problem 6. Evaluate $\frac{zdz}{(9-z^2)(z+i)}$ where C is the circle $|z| = 2$ taken in the positive sense.

Solution. Let $f(z) = \frac{z}{9-z^2}$. Clearly $f(z)$, is analytic within and on C.

∴ By Cauchy's integral formula

$$\int_C \frac{zdz}{(9-z^2)(z+i)} = \int_C \frac{f(z)}{z+i} dz$$

$$= 2\pi i f(-i)$$

$$= 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

Exercises

1.Prove that $\int_C \frac{zdz}{z^2-1} = 2\pi i$ where C is the positively oriented circle $|z| = 2$.

2.Evaluate $\int_C \frac{dz}{z^2+4}$ where C is $|z - i| = 2$ in the positive sense.

3. Evaluate $\int_C \frac{e^z dz}{z^2+1}$ where C is the circle of radius 1 with centre at

   (i) $z = i$ and

   (ii) $z = -i$.

4. Evaluate $\int_C \frac{\cos \pi z}{z^2-1} dz$ where C is a rectangle with vertices at (i) $2 \pm i, -2 \pm i$ and (ii) $-i, 2-i, 2+i, i$.

5. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt} dz}{z^2+1} = \sin t$ if $t > 0$ and C is the circle $|z| = 3$.

6. Evaluate $\int_C \frac{e^{3z}dz}{z-\pi i}$ where C is the circle $|z-1|=4$.

7. Evaluate $\int_C \frac{\sin 3z}{z+\pi/2}$ where C is the circle $|z|=5$.

8. Evaluate $\int_C \frac{dz}{z-3-i}$ where C is

(i) the square bounded by the real and imaginary axis and the lines $x=1$ and $y=1$;

(ii) the rectangle bounded by the real and imaginary axes and the lines $x=4$ and $y=3$ described in the anti-clockwise direction.

9. Evaluate $\int_C \frac{dz}{z^2(z-1)}$ where C is

(i) $|z|=\frac{3}{4}$

(ii) $|z|=\frac{3}{2}$.

10. Evaluate $\int_C \frac{3z-1}{z^3-z}dz$ where C is (i) $|z|=\frac{1}{2}$ (ii) $|z|=2$.

11. Evaluate $\int_C \frac{dz}{z^3(z-1)}$ where C is (i) $|z|=2$ (ii) $|z-1|=\frac{1}{2}$.

12. Evaluate $\int_C \frac{dz}{z^2(z^2+4)}$ where C is

(i) $|z|=\frac{3}{2}$

(ii) $|z|=1$

(iii) $|z-2i|=3$

(iv) $|z+2i|=3$

(v) $|z|=3$.

13. Evaluate the integral $\int_C \frac{zdz}{z^4-1}$ where C is the circle $|z-2|=2$.

14. Evaluate $\int_C \frac{(z+4)dz}{z^2+2z+5}$ where C is

(i) $|z|=1$

(ii) $|z+1-i|=2$

(iii) $|z+1+i|=2$

15. Evaluate $\int_C \frac{zdz}{(9-z^2)(z+1)}$ where C is the circle $|z| = 2$.

16. Prove $\int_C \frac{dz}{z^2+1} = 0$ where C is the positively oriented circle $|z| = 2$.

Answers.

2. $\frac{\pi}{2}$ 3. (i) $\pi(\cos 1 + i\sin 1)$ (ii) $-\pi(\cos 1 - i\sin 1)$

4. (i) 0 (ii) $-\pi i$ 6. $-2\pi i$ 7. $2\pi i$ 8. (i) 0 (ii) $2\pi i$

9. (i) $-2\pi i$ (ii) 0 10. (i) $2\pi i$ (ii) 0

11. (i) $-2\pi i$ (ii) $2\pi i$ 12. (i) 0 (ii) 0 (iii) 2π (iv) -2π (v) 0

13. $\frac{\pi i}{2}$. 14.(i) 0 (ii) $\pi\left(3 + \frac{2i}{2}\right)$ (iii) $\frac{\pi(2i-3)}{2}$ 15. $-\frac{\pi i}{4}$.

3.4. Higher Derivatives:

In this section we shall prove that an analytic function has derivatives of all orders. It follows, in particular, that the derivative of an analytic function is again an analytic function.

Consider a function $f(z)$ which is analytic in a region D. Let $z \in D$. Let C be any circle with centre z such that the circle and its interior is contained in D. By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We now proceed to prove that $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$ and in general $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}} d\zeta$.

Theorem 1:

Let f be analytic inside and on a simple closed curve C.

Let z be any point inside C. Then $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$

Proof. By Cauchy's integral formula we have $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z} d\zeta$

$$\therefore \frac{f(z+h)-f(z)}{h} = \frac{1}{h(2\pi i)}\int_C \left(\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z}\right) d\zeta$$

$$= \frac{1}{h2\pi i}\int_C \left[\frac{hf(\zeta)}{(\zeta - z - h)(\zeta - z)}\right] d\zeta$$

$=\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)}$ ……..(1)

Now $\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)} - \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}$

$$= \int_C \left[\frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} - \frac{f(\zeta)}{(\zeta - z)^2}\right] d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta - z)}\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right) d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta - z)}\left[\frac{h}{(\zeta - z - h)(\zeta - z)}\right] d\zeta$$

$$= \text{h}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - h)} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2} = \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

$\therefore \frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2} = \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}$ ……(2)

Now, let M denote the maximum value of $|f(\zeta)|$ on C. Let L be the length of C and d be the shortest distance from z to any point on the curve C.

$\therefore$ For any point ζ on C we have

$|\zeta - z| \geq d$ and $|\zeta - z - h| \geq |\zeta - z| - |h| \geq d - |h|$

Hence $\left|\frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)}\right| \leq \frac{M}{d^2(d - |h|)}$

From (2) we get

$$\left|\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right| \leq \frac{|h|}{2\pi}\left(\frac{M}{d^2(d-|h|)}\right)$$

$$\therefore \lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right)=0$$

$$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}$$

$$\therefore f'(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}$$

Remark.

By using induction on n we can prove that for any positive integer n we have

$f^{(r)}(z)=\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta.$

Note. Thus, an analytic function has derivatives of all orders and the derivate of an analytic function is again analytic.?

Example 1:

$\int_C \frac{e^z}{z^n}dz=\frac{2\pi i}{(n-1)!}$ where C is the circle $|z|=1$.

Let $f(z)=e^z$. Clearly $f(z)$ is analytic and $f^{(n)}(z)=e^z$ for all n.

By the formula for higher derivatives

$$\int_C \frac{e^z}{z^n}dz=\int_C \frac{e^z}{(z-0)^n}dz=\frac{2\pi i}{(n-1)!}e^0=\frac{2\pi i}{(n-1)!}$$

Example 2:

$\int_C \frac{\sin^2 z}{(z-\pi/6)^3}dz=\pi i$ where C is the circle $|z|=1$.

Solution:

Let $f(z)=\sin^2 z$. Then $f'(z)=2\sin z\cos z=\sin 2z$. $f''(z)=2\cos 2z$. Also $\pi/6$ lies inside C.

$$\begin{aligned}\therefore \int_C \frac{\sin^2 z}{(z-\pi/6)^3}dz &= \frac{2\pi i}{2!}f''(\pi/6)\\ &= \pi i(2\cos \pi/3)\\ &= \pi i\end{aligned}$$

Theorem 2: (Cauchy's inequality)

Let $f(z)$ be analytic inside and on the circle C with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on C. Then $\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$

Proof. We have $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$

$$\therefore \left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi}\left(\frac{M}{r^{n+1}}\right)(2\pi r) = \frac{n!\,M}{r^n}$$

Hence $\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$

Theorem 3: (Liouville's theorem)

A bounded entire function in the complex plane is constant.

Proof. Let $f(z)$ be à bounded entire function.

Since $f(z)$ is bounded there exists a real number M such that $|f(z)| \leq M$ for all z. Let z_0 be any complex number and $r > 0$ be any real number.

By Cauchy's inequality we have $|f'(z_0)| \leq \frac{M}{r}$.

Taking the limit as $r \to \infty$ we get $f'(z_0) = 0$.

Since z_0 is arbitrary $f'(z) = 0$ for all z in the complex plane.

$\therefore f(z)$ is a constant function.

Theorem 4:(Fundamental theorem of algebra)

Every polynomial of degree ≥ 1 has atleast one zero (root) in C.

Proof:

Let $f(z)$ be a polynomial of degree ≥ 1.

Suppose $f(z)$ has no zero in $\boldsymbol{C}$. Then $f(z) \neq 0$ for all z.

Further $f(z)$ is an entire function in the complex plane.

$\therefore \frac{1}{f(z)}$ is also an entire function. Also as $z \to \infty, f(z) \to \infty$.

$\therefore \frac{1}{f(z)} \to 0$ as $z \to \infty$.

$\therefore \frac{1}{f(z)}$ is a bounded function.

Hence by Liouville's theorem $\frac{1}{f(z)}$ is a constant function.

$\therefore f(z)$ is a constant function and hence it is a polynomial of degree zero which is a ontradiction.

Hence $f(z)$ has at least one root in C.

Hence the theorem.

Theorem 5: (Morera's theorem)

If $f(z)$ is continuous in a simply connected domain D and if $\int_C f(z)dz = 0$ for every imple closed curve C lying in D then $f(z)$ is analytic in D.

(This theorem is the converse of Cauchy's theorem)

Proof:

By corollary 1 of 6.2 there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Also we know the derivative of an analytic function is an analytic function.

Hence $F'(z)$ is analytic in D.

$\therefore f(z)$ is analytic in D.

Solved Problems

Problem 1.

Evaluate $\int_C \frac{\sin z}{(z-\pi/2)^2} dz$ where C is the circle $|z| = 2$.

Solution:

Let $f(z) = \sin z$. Hence $f'(z) = \cos z$. Also $\pi/2$ lies inside $|z| = 2$.

$$\text{Hence } \int_C \frac{\sin z dz}{(z-\pi/2)^2} = 2\pi i f'(\pi/2)$$
$$= 2\pi i(\cos \pi/2)$$
$$= 0$$

Problem 2:

Evaluate $\int_C \frac{z^3 dz}{(2z+i)^3}$ where C is the unit circle.

Solution:

$\int_C \frac{z^3\,dz}{(2z+i)^3} = \frac{1}{8}\int_C \frac{z^3\,dz}{(z+i/2)^3}$

Let $f(z) = z^3$. Then $f'(z) = 3z^2$ and $f''(z) = 6z$

Also $-\frac{i}{2}$ lies inside C.

Hence $\int_C \frac{z^3\,dz}{(2z+i)^3} = \frac{1}{8}\left(\frac{2\pi i}{2!}\right) f''\left(-\frac{i}{2}\right)$

$$= \frac{2\pi i}{16}(-3i)$$
$$= \frac{3\pi}{8}$$

Problem 3:

Evaluate $\int_C \frac{(e^z + z\sinh z)dz}{(z-\pi i)^2}$ where C is the circle $|z| = 4$.

Solution:

Let $f(z) = e^z + z\sinh z$

Therefore $f'(z) = e^z + z\cosh z + \sinh z$

Also πi lies inside C.

$$\text{Hence} \int_C \frac{f(z)}{(z-\pi i)^2}dz = 2\pi i f'(\pi i)$$
$$= 2\pi i\left[e^{\pi i} + \pi i\cosh \pi i + \sinh \pi i\right]$$
$$= 2\pi i(-1 - \pi i)$$
$$= -2\pi i(1 + \pi i)$$

Problem 4:

Show that when f is analytic within and on a simple closed curve C and z_0 is not on C then $\int_C \frac{f'(z)dz}{z-z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2}$

Solution:

Case i. Suppose z_0 is in the exterior of C. Then both $\frac{f(z)}{(z-z_0)^2}$ and $\frac{f'(z)}{z-z_0}$ are analytic inside and on C.

$\therefore$ By Cauchy's theorem $\int_C \frac{f'(z)dz}{z-z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2} = 0$

Case ii. z_0 lies in the interior of C.

Then by Cauchy's integral formula, $\int_C \frac{f'(z)}{(z-z_0)} dz = 2\pi i f'(z_0)$.

Also by the formula for higher derivatives $\int_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0)$.

Hence $\int_C \frac{f'(z)}{z-z_0} dz = \int_C \frac{f(z)}{(z-z_0)^2} dz$.

Problem 5:

Let the function $f(z) = u(x,y) + iv(x,y)$ be continuous in a closed ounded region D and let it be analytic and not constant in the interior of D. Show that re function $u(x,y)$ reaches its maximum value on the boundary of D and never in the merior of D.

Solution:

Consider the function $e^{f(z)}$. Since $f(z)$ is continuous in a closed bounded egion D and nonconstant in the interior of D, $e^{f(z)}$ is also continuous in the closed sounded region D and analytic and nonconstant in the interior of D.

Now, the maximum value of $\left|e^{f(z)}\right|$ is attained only at a boundary point of D.

But $\left|e^{f(z)}\right| = e^{u(x,y)}$

$\therefore$ Maximum value $e^{u(x,y)}$ is attained only at a boundary point of D.

$\therefore$ Maximum value of $u(x,y)$ is attained only at a boundary point of D.

Problem 6:

Evaluate $\int_C \frac{\sin 2z dz}{(z-\pi i/4)^4}$ where C is $|z| = 1$.

Solution:

Let $f(z) = \sin 2z$. Since $f(z)$ is analytic and $\pi i/4$ lies inside C.

$$\therefore \int_C \frac{\sin 2z}{(z-\pi i)^4} = \frac{2\pi i}{3!} f'''\left(\frac{\pi i}{4}\right)$$

Now $f'(z) = 2\cos 2z, f''(z) = -4\sin 2z; f'''(z) = -8\cos 2z$.

Hence $f'''(\pi i/4) = -8\cos(\pi i/2)$

$$= -8\cosh(\pi/2)$$

$$\therefore \int_C \frac{\sin z}{(z-\pi i)^4} dz = -\frac{8\pi i}{3}\cosh(\pi/2)$$

Problem 7:

Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 2$.

Solution:

Let $f(z) = e^{2z}$. Clearly $f(z)$ is analytic and

$f'(z) = 2e^{2z}; f''(z) = 4e^{2z}; f'''(z) = 8e^{2z}$

By the formula for higher derivatives

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \left(\frac{2\pi i}{3!}\right) f'''(-1)$$

$$= \left(\frac{2\pi i}{6}\right)(8e^{-2})$$

$$= \frac{i8\pi e^{-2}}{3}$$

Problem 8:

Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2} dz$ where C is $|z| = 3$.

Solution. $\frac{1}{(z+2)(z+1)^2} = \frac{(z+2)-(z+1)}{(z+2)(z+1)^2}$

$$= \frac{1}{(z+1)^2} - \frac{1}{(z+2)(z+1)}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z+2}$$

$$\int_C \frac{e^z}{(z+2)(z+1)} dz = \int_C \frac{e^z}{z+2} dz - \int_C \frac{e^z}{z+1} dz + \int_C \frac{e^z}{(z+1)^2} dz \quad \ldots\ldots\ldots (1)$$

We note that $z = -2, -1$ lie in the interior of C.

Let $f(z) = e^z$. It is analytic in C. Also $f'(z) = e^z$.

By Cauchy's integral formula

$$\int_C \frac{e^z}{z+2}dz = 2\pi i f(-2) = 2\pi i e^{-2}.$$

$$\int_C \frac{e^z}{z+1}dz = 2\pi i f(-1) = 2\pi i e^{-1}.$$

$$\int_C \frac{e^z}{(z+1)^2}dz = \left(\frac{2\pi i}{1!}\right) f'(-1) = 2\pi i e$$

$\therefore$ From (1) $\displaystyle\int_C \frac{e^z}{(z+2)(z+1)^2}dz = 2\pi i[e^{-2} - e^{-1} + e^{-1}]$

$$= 2\pi i e^{-2}$$

UNIT IV

Sequence and Series–Power Series–Taylor's series–Laurent series–Zeros of an Analytic function –Singularities.

Chapter 4: Sections -4.1 to 4.7

4. Power Series

Introduction

In this chapter we introduce the concept of power series and prove that there exists a circle of convergence associated with any power series. The power series converges within its circle of convergence, analytic and its derivative can be obtained by term wise differentiation. We also define some standard elementary functions such as exponential function, trigonometric function, hyperbolic function and logarithmic function in terms of power series.

4.1. Sequences and series

We assume that the reader is familiar with the concept of sequences and series of real numbers. This concept can naturally be extended to complex numbers also. We briefly give the required definitions and state without proof the theorems which are used subsequently.

Definition:

A sequence of complex numbers (z_n) is said to converge to a complex number z if given $\varepsilon > 0$ there exists a positive integer n_0 such that $|z_n - z| < \varepsilon$ for all $n \geq n_0$. In this case we write $(z_n) \rightarrow z$ or $\lim_{n\to\infty} z_n = z$.

Theorem 1:

Let (z_n) be a sequence of complex numbers. Let $z_n = x_n + iy_n$ and $z = x + iy$. Then $\lim_{n\to\infty} z_n = z$ if and onty if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Proof:

Let $(z_n) \to z$. Then given $\varepsilon > 0$ there exists a positive integer n_0 such that $|z_n - z| < \varepsilon$ for all $n \geq n_0$.

Now $|\text{Re}(z_n - z)| \leq |z_n - z|$

$\therefore \ |x_n - x| \leq |z_n - z|$

$\therefore \ |x_n - x| < \varepsilon$ for all $n \geq n_0$.

$\therefore \ \lim_{n\to\infty} x_n = x$.

Similarly $\lim_{n\to\infty} y_n = y$.

Conversely let $(x_n) \to x$ and $(y_n) \to y$.

$\therefore$ There exist positive integers n_1 and n_2 such that $|x_n - x| < \varepsilon/2$ for all $n \geq n_1$ and $|y_n - y| < \varepsilon/2$ for all $n \geq n_2$.

Let $n_0 = \max|n_1, n_2|$

$$\begin{aligned}\text{Now } |z_n - z_0| &= |x_n - x + i(y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< (\varepsilon/2) + (\varepsilon/2) = \varepsilon \text{ for all } n \geq n_0\end{aligned}$$

$\therefore \ \lim_{n\to\infty} z_n = z_1$

Definition:

Let (a_n) be a sequence of real numbers. If (a_n) is not bounded above then we define its upper limit to be ∞. Let (a_n) be a sequence which is bounded above. A real number u is called the upper limit of the sequence (a_n) if given $\varepsilon > 0$

(i) there exists $n_0 \in$ N such that $a_n < u + \varepsilon$ for all $n \geq n_0$

(ii) there exist infinitely many terms of the sequence (a_n) such that $a_n > u - \varepsilon$.

The upper limit of (a_n) is denoted by $\overline{\lim} a_n$ or lim sup a_n.

Similarly we can define the lower limit of any sequence of real numbers and is denoted by $\underline{\lim} a_n$ or lim infa_n.

Definition: Let $(z_n) = z_1, z_2, \dots \dots, z_n, \dots$ be a sequence of complex numbers. Then the formal expression $z_1 + z_2 + \cdots \dots + z_n + \cdots$ is called an infinite series of complex numbers and is denoted by $\sum_{n=1}^{\infty} z_n$ or simply $\sum z_n$.

Let $s_1 = z_1; s_2 = z_1 + z_2; s_3 = z_1 + z_2 + z_3; \ldots; s_n = z_1 + z_2 + \cdots + z_n$. Then (s_n) is called the **sequence of partial sums** of the given series $\sum z_n$. The series $\sum z_n$ is said to **converge, or diverge** according as the sequence of partial sums (s_n) converges or diverges.

If $(s_n) \to s$ we say that $\sum a_n$ converges to the sum s. We state without proof the following tests for convergence of series.

Comparison test:

Let $\sum c_n$ be a convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $a_n \le c_n$ for all $n \ge m$ then $\sum a_n$ is also convergent.

Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in \mathbf{N}$ such that $a_n \ge d_n$ for all $n \ge m$ then $\sum a_n$ is also divergent.

Result:

The harmonic series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges of $p \le 1$. Ratio test.

Let $\sum a_n$ be series of positive terms.

Then $\sum a_n$ converges if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Cauchy's Root test. Let $\sum a_n$ be a series of positive terms. Then $\sum a_n$ is convergent if $\lim\limits_{n\to\infty} a_n^{1/n} < 1$ and divergent if $\lim\limits_{n\to\infty} a_n^{1/n} > 1$.

pefinition. A series $\sum z_n$ is said to be absolutely convergent if $\sum |z_n|$ is convergent.

Example 1:

The series $\sum \frac{i^n}{n^2}$ is absolutely convergent, for $\sum \left|\frac{i^n}{n^2}\right| = \sum \frac{1}{n^2}$ which is convergent.

Example 2: The series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent for $\sum \left|\frac{(-1)^n}{n}\right| =$

$\sum \frac{1}{n}$ which is divergent. However, the given series is convergent.

Theorem 2:

Any absolutely convergent series is convergent.

However the converse is not true. For example, the series $\sum \frac{(-1)^n}{n}$ is convergent but not absolutely convergent.

Note. Since $\sum |a_n|$ is a series of positive terms the tests mentioned above can be used to test the absolute convergence of a given series.

4.2. Sequences and Series of Functions

Definition:

Let (f_n) be a sequence of complex functions defined in a region D. Suppose that for each $z \in D$ the sequence of complex numbers $(f_n(z))$ is convergent. Then we define $\lim_{n\to\infty} f_n(z) = f(z)$. We say that (f_n) converges pointwise to f on D and f is called the pointwise limit of (f_n).

More explicitly the sequence (f_n) converges pointwise to f if given $z \in D$ and $\varepsilon > 0$ there exists a positive integer n_0 such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq n_0$, Observe that n_0 in general depends both on ε and z.

Definition:

Let (f_n) be a sequence of complex functions defined in a region D. Let f be another complex function defined in D. The sequence (f_n) is said to converge uniformly to f in D if given $\varepsilon > 0$ there exists a positive integer n_0 such that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq n_0$ and for all $z \in D$.

Remark. It is immediate from the definition that

uniform convergence $\Rightarrow$ pointwise convergence.

Definition: Let (f_n) be a sequence of complex functions defined in a region D. The series $\sum_{n=1}^{\infty} f_n$ is said to converge pointwise to a function f defined in D if the sequence of functions (s_n) conveges to f pointwise on D

where $s_n = f_1 + f_2 + \cdots + f_n$.

The series $\sum_{v=1}^{\infty} f_n$ is said to converge uniformly to a function f in D if the sequence of functions (s_n) converges uniformly to f in D.

Weier strass M-Test. Let $\sum\ f_n$ be a series of complex functions defined in a region D. Let $\sum\ M_n$ be a convergent series of positive real numbers. If $|f_n(z)| \leq M_n$ for allog and for all $<\in D$, then the series $\sum\ f_n$ converges uniformly on D.

4.3. Power Series

Definition:

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

where the coefficients $a_0, a_1, \ldots, a_n$ ..., and the variable z are complex numbers is called a power series.

More generally any series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ where the coefficients a_n, z_0 and the variable z are complex numbers is called a power series about z_0.

We shall prove results for power series for the special case $z_0 = 0$. The corresponding results and their proofs for the general case are essentially similar.

Example:

Consider the power series (geometric series)

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \ldots + z^{n-1} + \cdots$$

The partial sum of the series is given by $s_n = 1 + z + z^2 + \cdots + z^{n-1}$

$$= \frac{z^n - 1}{z - 1}.$$

When $|z| < 1$ the sequence $(z^n) \to 0$ as $n \to \infty$, and hence $(s_n) \to \frac{1}{1-z}$.

Hence the power series converges when $|z| < 1$.

When $|z| \geq 1$ the sequence (z^n) diverges and hence the power series diverges.

Thus, the given power series converges if $|z| < 1$ and diverges if $|z| \geq 1$.

We shall prove in general that every power series converges inside a circle and diverges outside the same circle.

Theorem 1:

Let $\sum_{n=0}^{\infty} a_n z^n$ be a given power series. Then there exists a number R such that $0 \leq R \leq \infty$, called the radius of convergence of the power series with the following properties.

(i) The series converges absolutely for every z with $|z| < R$.

(ii) If $0 < \rho < R$ the convergence is uniform in $|z| \leq \rho$.

(iii) If $|z| > R$ the terms of the series are unbounded and hence the series diverges.

Proof:

Let $R = \frac{1}{\overline{\lim}|a_n|^{1/n}}$.

We shall prove that for this choice of R the assertions of the theorem are true.

Case(i). $R = 0$

In this case we need to prove that the series diverges for all $z \neq 0$.

$R = 0 \Rightarrow \overline{\lim}|a_n|^{1/n} = \infty$.

Hence the sequence $\left(|a_n|^{1/n}\right)$ is not bounded above.

$\therefore |a_n|^{1/n} > \frac{2}{|z|}$ for an infinite number of values of n so that $|a_n z^n| > 2^n$ for an infinite number of values of n.

Hence the terms of the series are unbounded and the series is divergent.

Case(ii). $R = \infty$. We claim that the power series converges for all z. The serits obviously converges when $z = 0$. Let z be any non zero complex number. $R = \infty \Rightarrow \overset{\text{e}}{\lim}|a_n|^{1/n} = 0$.

$\therefore$ Given $\varepsilon > 0$ there exists a positive integer n_0 such that $|a_n|^{1/n} < \varepsilon$ for all

$n \geq i_0$.

In particular $|a_n|^{1/n} < \frac{1}{2|z|}$ for all $n \geq n_0$.

$\therefore$ $|a_n z''| < \frac{1}{2^n}$ for all $n \geq n_0$. Since $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent series of positive terms by comparison test $\sum_{n=0}^{\infty} |a_n z^n|$ is convergent.

$\therefore$ $\sum \ a_n z^n$ is absolutely convergent for all z.

Case (iii) $0 < R < \infty$.

By Cauchy's root test $\sum \ a_n z^n$ is absolutely convergent if $\overline{\lim} |a_n z^n|^{1/n} < 1$

Now,
$$\begin{aligned} \overline{\lim} |a_n z^n|^{1/n} &= \overline{\lim} |a_n|^{1/n} |z| \\ &= |z| \overline{\lim} |a_n|^{1/n} \\ &= |z| \times \frac{1}{R} \end{aligned}$$

Hence $\overline{\lim} |a_n z_n|^{1/n} < 1$ if $|z| < R$.

$\therefore$ The power series converges absolutely if $|z| < R$.

Similarly the series is divergent if $|z| > R$.

Now let $0 < \rho < R$. We claim that the given power series converges uniformly if $|z| \leq \rho$.

Choose a real number ρ_0 such that $\rho < \rho_0 < R$.

$\therefore$ $\frac{1}{\rho_0} > \frac{1}{R} = \Pi_{\text{im}} |a_n|^{1/n}$. Hence $\Pi |a_n|^{1/n} < \frac{1}{\rho_0}$.

By definition of upper limit there exists a positive integer n_0 such that

$a_{n_1}\Big|^{1/n} < \frac{i}{\rho_0}$ for all $n \geq n_0$.

$\therefore$ $|a_n z^n| < \left(\frac{|z|}{\rho_0}\right)^n$ for all $n \geq n_0$.

Since $|z| \leq \rho$ we have $|a_n z^n| < \left(\frac{\rho}{\rho_0}\right)^n$ for all $n \geq n_0$.

Now, since $\rho < \rho_0$ we have $\frac{\rho}{\rho_0} < 1$.

$\therefore$ $\sum \ \left(\frac{\rho}{\rho_0}\right)^n$ is convergent series of constant terms.

$\therefore$ By Weierstrass M-test $\sum a_n z^n$ is uniformly convergent in $|z| \leq \rho$.

Remark 1. If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists then $\lim_{n\to\infty} |a_n|^{1/n}$ exists and the two limits are equal. Hence if $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists then $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Remark 2. There is no information about the behavior of the power series on the circle of convergence. In fact, the behavior of the power series on the circle of convergence can be of any type as shown in the following examples.

Example 1:

Consider the power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

Here $a_n = \dfrac{1}{n^2}$ and $a_{n+1} = \dfrac{1}{(n+1)^2}$.

Now,
$$\begin{aligned} R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \\ &= 1 \end{aligned}$$

Radius of convergence $R = 1$.

Now, if $|z| = 1$, then $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is absolutely convergent at every point on the circle of convergence.

Example 2:

Consider the geometric series $\sum_{n=1}^{\infty} z^n$. Obviously $R = 1$ and the series diverges at every point on the circle of convergence.

Example 3:

Consider the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$

Here $a_n = \frac{1}{n}$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

When $z = 1$ the power series becomes $\sum_{n=1}^{\infty}\frac{1}{n}$ which is a divergent series of positive terms.

When $z = -1$ the power series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which is a convergent series.

Thus the power series is convergent at some point on the circle of convergence and divergent at some other point on the circle of convergence.

Exercises:

1. Find the radii of convergence for the following series.

   (i) $\sum \frac{z^n}{n!}$ (ii) $\sum \frac{(n+1)z^n}{(n+2)(n+3)}$ (iii) $\sum (4+3i)^n z^n$

   (iv) $\sum \frac{n!z^n}{n^n}$ (v) $\sum \left(1+\frac{1}{n}\right)^{n^2} z^n$ (vi) $\sum \left(1-\frac{1}{n}\right)^{n^2} z^n$

   (vii) $\sum \left(\frac{n\sqrt{2}+i}{1+2in}\right) z^n$ (viii) $\sum z^{n!}$

2. Prove that $1+\frac{a-b}{1+c}+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\cdots$ has unit radius of convergence.

3. Nind the radius of convergence of the series

   $\frac{1}{2}2+\frac{1\cdot 3}{2\cdot 5}2^2+\frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}2^3+\cdots\ldots$

Answers

1. (1) ∞ (ii) 1 (iii) 5 (iv) e (v) $\frac{1}{e}$ (vi) e (vii) 1 (viii) I

3. $\frac{3}{2}$

Theorem 2:

Every power series represents an analytic function at all points within the circle of convergence and its derivative can be obtained by term wise differentiation of the given power series.

Proof:

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ………. (1)

Let R denote the radius of convergence of the power series (1) and let $R > 0$.

We know that $\frac{1}{R} = \overline{\lim}|a_n|^{1/n}$ ………. (2)

Consider the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ …………(3)

which is obtained by termwise differentiation of (1).

Let R_1 denote the radius of convergence of the power series (3).

We claim that $R = R_1$.

Since multiplication by z does not alter the radius of convergence of the power series it follows that $R_1 = \frac{1}{\overline{\lim}|n a_n|^{1/n}}$

$$= \frac{1}{\overline{\lim} n^{1/n}|a_n|^{1/n}}$$

$$= \frac{1}{\overline{\lim}|a_n|^{1/n}} \left(\text{since } \lim_{n\to\infty} n^{1/n} = 1 \right)$$

$$= R$$

Thus $R = R_1$ and the derived series has the same radius of convergence as the given power series.

Now, let $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$

We claim that $f'(z) = g(z)$.

Let $s_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ and

$R_n(z) = \sum_{k=n}^{\infty} a_k z^k$ so that $f(z) = s_n(z) + R_n(z)$.

Then $s_n'(z) = a_1 + 2a_2 z + \cdots \ldots + (n-1)a_{n-1} z^{n-2}$ which is a partial sum of the derived series.

Hence $\lim_{n\to\infty} s_n'(z) = g(z)$ in $|z| < R$.

Let z_0 be any point in $|z| < R$.

Now $\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right|$

$$= \left| \frac{s_n(z) + R_n(z) - s_n(z_0) - R_n(z_0)}{z - z_0} - g(z_0) \right.$$

$$= \left| \frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0) + \frac{R_n(z) - R_n(z_0)}{z - z_0} + s_n'(z_0) - g(z) \right|$$

$$\leq \left| \frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0) \right| + \left| \frac{R_n(z) - R_n(z_0)}{z - z_0} \right| + |s_n(z_0) - g(z)| \cdots \ldots (5)$$

Since $\lim\limits_{n\to\infty} s_n(z_0) = g(z_0)$ there exists a positive integer n_0 such that

$$|s_n(z_0) - g(z_0)| < \varepsilon/3 \text{ for all } n/\geq n_0 \quad ..(6)$$

Also $\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| = \sum_{k=n}^{\infty} \left|\frac{a_k(z^k - z_0^k)}{z - z_0}\right|$

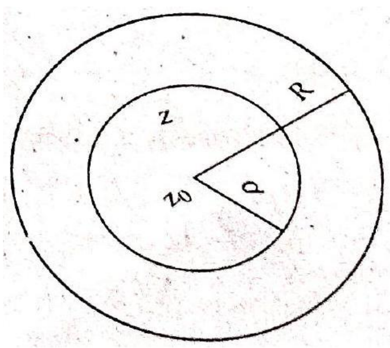
$$= \left| \sum_{k=n}^{\infty} a_k\left(z^{k-1} + z^{k-2}z_0 + \cdots + zz_0^{k-2} + z_0^{k-1}\right) \right|$$

$$\leq \sum_{k=n}^{\infty} |a_k|(|z|^{k-1} + |z|^{k-2}|z_0| + \cdots + |z_0|^{k-1})$$

$$\leq \sum_{k=n}^{\infty} |a_k| k\rho^{k-1} \qquad \ldots\ldots\ldots(7)$$

where ρ is a real number chosen in such a way that $|z|, |z_0| < \rho < R$.

The expression in the right side of (7) is the remainder after n terms of the convergent series $\sum_{n=1}^{\infty} na_n\rho^{n-1}$ and hence tends to 0 as $n \to \infty$.



Hence, we can find a positive integer n, such that

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| < \frac{\varepsilon}{3 \text{ for all } n} \geq n_1 \quad \dots\dots\dots\dots\dots (8)$$

Now choose a fixed positive integer n such that $n > n_0, n_1$.

By the definition of the derivative we can find $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow \left|\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0)\right| < \frac{\varepsilon}{3} \qquad \dots\dots\dots.(9)$$

Using (6), (8) and (9) in (5) we see that

$$0 < |z - z_0| < \delta \Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)\right| < \varepsilon.$$

Hence $f'(z_0) = g(z_0)$.

Since z_0 is an arbitrary point in $|z| < R$, we have $f'(z) = g(z)$ in $|z| < R$.

Remark: If the power series $f(z) = \sum_{n=0}^{\infty} a_n z_n$ has a non zero radius of convergence, by repeated application of the above theorem we see that it has derivatives of all orders. Hence, we have

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots .. + a_n z^n + \cdots ..$$
$$f'(z) = a_1 + 2a_2 z + \cdots ... + n a_n z^{n-1} + \cdots ..$$
$$f''(z) = 2a_2 + 6a_3 z + 12 a_4 z^2 + \cdots ... + n(n-1) z^{n-2} + \cdots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.$$
$$f^{(n)}(z) = n!\, a_n + (n+1)!\, a_{n+1} z + \cdots ..$$

Hence $f^{(n)}(0) = n!\, a_n$ so that $a_n = \frac{f^{(n)}(0)}{n!}$ and the power series becomes

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \cdots + \frac{f^{(n)}(0)}{n!} z^n + \cdots$$

which is the Maclaurin's series for $f(z)$.

Hence every power series is the Maclaurin's series of the analytic function which it defines within its circle of convergence.

Series Expansions

Introduction:

Intis chapter we consider the problem of representing a given function as a power series. We prove that if a function is analytic at a point z_0 then it can be expanded as gop wer series called Taylor's series consisting of non-negative powers of $z - z_0$ and ne expansion is valid in some neighbourhood of z_0. We also prove that a function $f(z)$ fuch is analytic in an annular region $a < |z - z_0| < b$ can be expanded as a series called Laurent's series consisting of positive and negative powers of $z - z_0$. We also introduce the concept of singular points of a function and classify the singular points and discuss the behaviour of the function in the neighbourhood of a singularity.

4.4. Taylor's Series

Theorem 1: (Taylor's Theorem)

Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots\cdots$$

The expansion is valid in the largest open disc with center z_0 contained in D.

Proof:

Let $r > 0$ be such that the disc $|z - z_0| < r$ is contained in D.

Let $0 < r_1 < r$. Let C_1 be the circle $|z - z_0| = r_1$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \ldots\ldots\ldots.(1)$$

Also, by theorem on higher derivatives, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}} \ldots\ldots\ldots..(2)$$

Now,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots\cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{\mathrm{n}} + + \frac{\left(\frac{z - z_0}{\zeta - z_0}\right)^2}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right]$$

(using the identify $\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}$)

$$= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \cdots \ldots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n(\zeta - z)}$$

Now, multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1) and (2) we get

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$
$$\cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n \qquad \ldots\ldots\ldots\ldots.(3)$$

where $R_n = \frac{(z-z_0)^n}{2\pi i}\int_a \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-z_0)^n}$.

Here ζ hes on C_1 and z lies in the interior of C_1 so that $|z - z_0| = r_1$ and $|z - z_0| < n$.

$$\therefore |k - z| = |(\zeta - z_0) - (z - z_0)| \geq |5 - z_0| - |k - z_0| = r_1 - |z - z_0|.$$

$$\frac{1}{|\,k - z_1} \leq \frac{1}{r_1 - |z - z_0|}.$$

Let M denote the maximum value of $|f(z)|$ on C_1.

Then $|R_n| = \frac{|z-z_0|}{2\pi}\frac{M(2\pi r_1)}{(r_1-|z-z_0|)r_1^{\mathrm{n}}}$ (by theorem 6.2)

$$= \frac{M|z - z_0|}{(r_1 - |z - z_0|)}\left(\frac{|z - z_0|}{r_1}\right)^{n-1}$$

Also $\left|\frac{z-z_0}{r_1}\right| < 1$. Hence $\lim_{n\to\infty} R_n = 0$.

$\therefore$ Taking limit as $n \to \infty$ in (3) we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

Note 1.

The above series is called the Taylor series of $f(z)$ about the point z_0. Thus if $f(z)$ is analytic at a point z_0 then $f(z)$ can be represented as a Taylor's series about z_0, which is a series in non negative powers of $z - z_0$. The expansion is valid in some neighbourhood of z_0.

Note 2.

The Taylor series expansion of $f(z)$ about the point zero is called the Maclaurin's series. Thus, the Maclaurin's series of $f(z)$ is given by

$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^n}{n!}f^{(n)}(0) + \cdots$$

Example 1:

The Taylor's series for $f(z) = \frac{1}{z}$ about $z = 1$ is given by

$$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z - 1) + \frac{f''(1)}{2!}(z - 1)^2 + \frac{f'''(1)}{3!}(z - 1)^3 + \cdots$$

$$\text{Now, } f(z) = \frac{1}{z} \Rightarrow f(1) = 1$$

$$f'(z) = -\frac{1}{z^2} \Rightarrow f'(1) = -1$$

$$f''(z) = \frac{2}{z^3} \Rightarrow f''(1) = 2$$

$$f'''(z) = -\frac{6}{z^4} \Rightarrow f'''(1) = -6$$

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \cdots$$

This expansion is valid in the disc $|z-1|<1$.

Similarly the Taylor's series for $f(z)=\frac{1}{z}$ about $z=i$ is given by

$$\frac{1}{z}=\frac{1}{i}-\frac{z-i}{i^2}+\frac{(z-i)^2}{i^3}-\frac{(z-i)^3}{i^4}+\cdots$$

and the expansion is valid in the disc $|z-i|<1$. (verify)

Example 2:

Let $f(z)=e^z$.

Then $f^{(n)}(z)=e^z$ for all n and hence $f^{(n)}(0)=1$.

Hence the Maclaurin's series for e^z is given by

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots$$

and the expansion is valid in the entire complex plane.

Maclaurin's series expansion of some of the standard functions are given below.

1. $e^{-z}=1-\frac{z}{1!}+\frac{z^2}{2!}-\cdots+(-1)^n\frac{z^n}{n!}+\cdots\ (|z|<\infty)$
2. $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots+(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}+\cdots\ (|z|<\infty)$
3. $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots+(-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!}+\cdots\ (|z|<\infty)$
4. $\sinh z=\frac{1}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots+\frac{z^{2n-1}}{(2n-1)!}+\cdots\ (|z|<\infty)$
5. $\cosh z=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots+\frac{z^{2n}}{(2n)!}+\cdots\ (|z|<\infty)$
6. $\frac{1}{1+z}=1-z+z^2-z^3+\cdots+(-1)^nz^n+\cdots\ (|z|<1)$
7. $\frac{1}{1-z}=1+z+z^2+z^3+\cdots+z^n+\cdots\ (|z|<1)$
8. $\log(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots(-1)^{n-1}\frac{z^n}{n}+\cdots\ (|z|<1)$
9. $\log(1-z)=-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots-\frac{z^n}{n}-\cdots\ (|z|<1)$.

Solved problems

Problem 1:

Expand $\cos z$ into a Taylor's series about the point $z = \pi/2$ and determine the region of convergence.

Solution:

Let $f(z) = \cos z$

The Taylor's series for $f(z)$ about $z = \pi/2$ is

$$f_{(z)} = f(\pi/2) + \frac{(z-\pi/2)}{1!} f'(\pi/2) + \frac{(z-\pi/2)^2}{2!} f''(\pi/2) + \frac{(z-\pi/2)^3}{3!} f'''(\pi/2) + \cdots$$

Now $f(z) = \cos z$. Hence $f(\pi/2) = 0$.

$f'(z) = -\sin z$. Hence $f'(\pi/2) = -1$
$f''(z) = -\cos z$. Hence $f''(\pi/2) = 0$
$f'''(z) = \sin z$. Hence $f'''(\pi/2) = 1$

∴ The Taylor's series for $\cos z$ about $z = \pi/2$ is

$$\cos z = -\frac{(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

The expansion is valid throughout the complex plane.

Problem 2:

Expand $f(z) = \sin z$ in a Taylor's series about $z = \pi/4$ and determine the region of convergence of this series.

Solution:

The Taylor's series for $f(z)$ about $z = \pi/4$ is

$$f(z) = f(\pi/4) + \frac{(z-\pi/4)}{1!} f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \cdots$$

Here $f(z) = \sin z$. Hence $f(\pi/4) = \frac{1}{\sqrt{2}}$:

$$f'(z) = \cos z. \text{ Hence } f'(\pi/4) = \frac{1}{\sqrt{2}}.$$
$$f''(z) = -\sin z. \text{ Hence } f''(\pi/4) = -\frac{1}{\sqrt{2}}.$$
$$f'''(z) = -\cos z. \text{ Hence } f'''(\pi/4) = -\frac{1}{\sqrt{2}}.$$

The Taylor's series for $\sin z$ about $z = \pi/4$ is

$$\sin z = \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right) - \frac{(z-\pi/4)^2}{2!}\left(\frac{1}{\sqrt{2}}\right) + \cdots\cdots$$
$$= \frac{1}{\sqrt{2}}\left[1 + \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right]$$

The expansion is valid in the entire complex plane.

Problem 3:

Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series

(i) about the point $z = 0$.

(ii) about the point $z = 1$. Determine the region of convergence in each case.

Solution.

(i) $f(z) = \frac{z-1}{z+1}$

$$\begin{aligned} &= (z-1)(1+z)^{-1} \\ &= (z-1)(1 - z + z^2 - z^3 + \cdots\cdots) \text{ if } |z| < 1 \\ &= (z - z^2 + z^3 - \cdots\cdots) - (1 - z + z^2 - z^3 - \cdots\cdots) \\ &= -1 + 2z - 2z^2 + 2z^3 + \cdots\cdots \end{aligned}$$

The region of convergence is $|z| < 1$.

(ii) $f(z) = \frac{z-1}{z+1}$

$= \frac{z-1}{(2+z-1)}$

$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$

$= \frac{z-1}{2}\left(1 + \frac{z-1}{2}\right)^{-1}$

$$= \frac{z-1}{2}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\cdots\right] \text{ if } \left|\frac{z-1}{2}\right| < 1$$

$$= \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} - \cdots\cdots$$

The region of convergence is given by $\left|\frac{z-1}{2}\right| < 1$ which is same as the circular disc $|z-1| < 2$.

Problem 4:

Show that

(i) $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ when $|z+1| < 1$.

(ii) $\frac{1}{z^2} = \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n$ when $|z-2| < 2$.

Solution:

(i)
$$\frac{1}{z^2} = \frac{1}{[1-(z+1)]^2}$$
$$= [1-(z+1)]^{-2}$$
$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \cdots\cdots \text{ if } |z+1| < 1$$
$$= 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n \text{ when } |z+1| < 1.$$

(ii) $\frac{1}{z^2} = \frac{1}{(z-2+2)^2}$

$$= \frac{1}{\left[2\left(1 + \frac{z-2}{2}\right)\right]^2}$$

$$= \frac{1}{4}\left(1 + \frac{z-2}{2}\right)^{-2}$$

$$= \frac{1}{4}\left[1 - 2\left(\frac{z-2}{2}\right) + 3\left(\frac{z-2}{2}\right)^2 - \cdots\right] \text{ if } \left|\frac{z-2}{2}\right| < 1$$

$$= \frac{1}{4} - \frac{1}{4} \times 2\left(\frac{z-2}{2}\right) + \frac{1}{4} \times 3\left(\frac{z-2}{2}\right)^2 - \cdots$$

$$= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n$$

Here the region of convergence is $\left|\frac{z-2}{2}\right| < 1$ which is the same as the circular disc$|z-2| < 2$.

Problem 5:

Expand ze^{2z} in a Taylor's series about $z = -1$ and deternine the region of convergence.

Solution:

Let $f(z) = ze^{2z}$

$$= ze^{2(z+1)}e^{-2}$$
$$= \frac{1}{e^2}\left[(z+1)e^{2(z+1)} - e^{2(z+1)}\right]$$
$$= \frac{1}{e^2}\left[(z+1)\left\{1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \cdots\right\}\right.$$
$$\left.-\left\{1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \cdots\right\}\right]$$
$$= \frac{1}{e^2}\left[\left\{(z+1) + \frac{2(z+1)^2}{1!} + \frac{2^2(z+1)^3}{2!} + \cdots\right\}\right.$$
$$\left.-\left\{1 + \frac{2(z+1)}{1!} + \frac{2^2(z+1)^2}{2!} + \cdots\right\}\right]$$
$$= \frac{1}{e^2}\left[-1 + \left(1 - \frac{2}{1!}\right)(z+1) + \left(\frac{2}{1!} - \frac{2^2}{2!}\right)(z+1)^2 + \left(\frac{2^2}{2!} - \frac{2^3}{3!}\right)(z+1)^3 + \cdots\right.$$

The expansion is valid throughout the complex plane.

Problem 6:

Find the Taylor's scries to represent $\frac{z^2-1}{(z+2)(z+3)}$ in $|z| < 2$

Solution:

By partial fractions

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z+2}-\frac{8}{z+3} \text{ (verify)}$$

$$= 1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$= \left(1+\frac{3}{2}-\frac{8}{3}\right)+\left(-\frac{3}{2^2}+\frac{8}{3^2}\right)z+\left(\frac{3}{2\cdot 2^2}-\frac{8}{3\cdot 3^2}\right)z^2+\cdots$$

$$= -\frac{1}{6}+\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{8}{3^{n+1}}-\frac{3}{2^{n+1}}\right)z^n$$

and the expansion is valid in $|z| < 2$.

Exercises.

1.Expand $\frac{1}{z}$ about $z = -1$ and $z = 2$ as Taylor's series, stating the region of convergence.

2.Show that $\frac{1}{z^2} = 1-2(z-1)+3(z-1)^2-4(z-1)^3+\cdots$ for all z in $|z-1| < 1$.

3.Expand $\frac{z}{z-3}$ as a Taylor's series about $z = 1$.

4.Find the Maclaurin's series for $\frac{1}{2-z}$. What is its radius of convergence.

5.Obtain the Taylor's series to represent $\frac{1}{(z+1)(z+3)}$ in $|z| < 1$.

6.Obtain the Taylor's series for $\frac{z}{z+2}$ about $z = 1$. State the region of validity.

7.Find the Taylor's series for ze^z about $z = 1$.

8.Show that $\sin z^2 = z^2-\frac{z^6}{3!}+\frac{z^{10}}{5!}-\frac{z^{14}}{7!}+\cdots$ for $|z| < \infty$.

Answers.

2.$\frac{1}{z} = -1-(z-1)-(z-1)^2+\cdots;\ |z+1| < 1$

$$\frac{1}{z} = \frac{1}{2} - \frac{z-2}{2^2} + \frac{(z-2)^2}{2^3} - \frac{(z-2)^3}{2^4} + \cdots ; |z-2| < 2$$

4. $-\frac{1}{2} - \frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{z-1}{2}\right)^n$ 5. $\sum_0^{\infty} \frac{z^n}{2^{n+1}}; 2$

6. $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[1 - \frac{1}{3^{n+1}}\right]z^n$

7. $\frac{1}{3} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{3^{n+1}}; |z-1| < 3$

8. $e\left[1 + \frac{2(z-1)}{1!} + \frac{3(z-1)^2}{2!} + \frac{4(z-1)^3}{3!} + \cdots\cdots\right]$

4.5 Laurent's Series

A series of the form $\sum_{n=1}^{\infty} \frac{b_n}{z_n}$ ....(1)

can be considered as an ordinary power series in the variable $\frac{1}{z}$. Hence if the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$ is r and $r < \infty$ the series $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$ converges in the region $|z| > r$.

The convergence is uniform in every region $|z| \geq \rho > r$ and the series represents an analytic function in $|z| > r$.

If the series (1) is combined with the usual power series we get a more general series of the form $\sum_{-\infty}^{\infty} a_n z$ .... (2)

This series is said to converge at a point if the part of the series consisting of the negative powers of z and the part of the series consisting of non-negative powers of z are separately convergent. We know that the series consisting of non-negative powers of z converges in a disc $|z| < r_2$ and the series consisting of negative powers of z converges in a region $|z| > r_1$. $\therefore$ If $r_1 < r_2$ the series represented by (2) converges in the region $r_1 < |z| < r_2$ and in this annulus region it represents an analytic function.

We shall now prove that the converse situation is also true. (i.e) any function which is analytic in a region containing the annulus

$r_1 < |z - z_0| < r_2$ can be represented in a series of the form

$\sum_{-\infty}^{\infty} a_n (z - z_0)^n$.

Theorem 1: (Laurent's theorem)

Let C_1 and C_2 denote respectively the concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ with $r_1 < r_2$. Let $f(z)$ be analytic in a region containing the circular annulus $r_1 < |z - z_0| < r_2$. Then $f(z)$ can be represented as a convergent series of positive and negative powers of $z - z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{-n+1}}$ and $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$.

Proof:

Let z be any point in the circular annulus $r_1 < |z - z_0| < r_2$.

we have, $f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{\zeta - z}$.

$$\therefore\ f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{z - \zeta} \ldots \ldots \ldots (1)$$

As in the proof of Taylor's theorem, we have

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 +$$

$$\cdots \cdots + a_{n-1} (z - z_0)^{n-1} + R_n(z) \ldots \ldots \quad (2)$$

where $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$ and

$$R_n(z) = \frac{(z - z_0)^n}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^n (\zeta - z)}$$

$$\frac{1}{z-\zeta}=\frac{1}{z-z_0+z_0-\zeta}$$

$$=\frac{1}{(z-z_0)-(\zeta-z_0)}$$

Now,

$$=\frac{1}{(z-z_0)\left[1-\frac{\zeta-z_0}{z-z_0}\right]}$$

$$=\frac{1}{z-z_0}\left[1+\left(\frac{\zeta-z_0}{z-z_0}\right)+\left(\frac{\zeta-z_0}{z-z_0}\right)^2+\cdots\cdots+\left(\frac{\zeta-z_0}{z-z_0}\right)^{n-1}\right.$$

$$\left.+\frac{\left(\frac{\zeta-z_0}{z-z_0}\right)^n}{1-\left(\frac{\zeta-z_0}{z-z_0}\right)}\right]$$

Multiplying by $\frac{f(\zeta)}{2\pi i}$ and integrating over C_1 we get

$$\int_{C_1}\frac{f(\zeta)d\zeta}{z-\zeta}=\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_{n-1}}{(z-z_0)^{n-1}}+S_n(z)\ \ldots\ldots\ldots..(3)$$

where $b_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}};S_n=\frac{1}{2\pi i(z-z_0)^n}\int_{C_1}\frac{f(\zeta)(\zeta-z_0)^n d\zeta}{z-\zeta}$

From (1), (2) and (3) we get

$$f(z)=a_0+a_1(z-z_0)+\cdots\cdots+a_{n-1}(z-z_0)^{n-1}$$
$$+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots\cdots+\frac{b_{n-1}}{(z-z_0)^{n-1}}+R_n(z)+S_n(z)\ \ldots\ldots\ldots(4)$$

The required result follows if we can prove that $R_n\to 0$ and $S_n\to 0$ as $n\to\infty$.

Now, if $\zeta\in C_1$ then $|\zeta-z_0|=r_1$ and

$$|z-\zeta|=|(z-z_0)-(\zeta-z_0)|\geq|z-z_0|-r_1.$$

If $\zeta\in C_2$ then $|\zeta-z_0|=r_2$ and

$$|\zeta-z|=|(\zeta-z_0)-(z-z_0)|\geq r_2-|z-z_0|$$

Now let M denote the maximum value of $|f(z)|$ in $C_1\cup C_2$.

Then $|R_n|\leq\frac{|z-z_0|^n}{2\pi}\frac{M(2\pi r_2)}{r_2^n(r_2-|z-z_0|)}$. (by theorem 2)

$$\leq\frac{M|z-z_0|}{(r_2-|z-z_0|)}\left(\frac{|z-z_0|}{r_2}\right)^{n-1}$$

Since $\frac{|z-z_0|}{r_2} < 1, R_n \to 0$ as $n \to \infty$.

Also $|S_n| \leq \frac{1}{|z-z_0|^n 2\pi} \frac{M r_1^n (2\pi r_1)}{(|z-z_0|-r_1)}$

$$\leq \frac{M r_1}{(|z-z_0|-r_1)} \left(\frac{r_1}{|z-z_0|}\right)^n$$

Since $\frac{r_1}{|z-z_0|} < 1, S_n \to 0$ as $n \to \infty$.

Hence, by taking limit $n \to \infty$ in (4) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

Hence the theorem.

Remark:

The formulae for the coefficients a_n and b_n in the Laurent's series expansion are given by $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$ .... (1)

$$\text{and } b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{-n+1}} \dots\dots..(2)$$

Since the integrands in the integrals of (1) and (2) are analytic functions of ζ throughout the annular region, any simple closed curve C in the annulus can be used as the path of integration in place of C_1 and C_2.

Hence Laurent's series can be written as

$$f(z) = \sum_{-\infty}^{\infty} A_n (z-z_0)^n, (r_1 < |z-z_0| < r_2)$$

where $A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$

Solved Problems

Problem:

Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$ about $z = 0$.

Solution:

$f(z) = z^2 e^{1/z}$.

Clearly $f(z)$ is analytic at all points $z \neq 0$.

Now, $$f(z) = z^2\left[1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right]$$
$$= z^2 + z + \frac{1}{2} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \cdots$$

This is the required Laurent's series expansion for $f(z)$ at $z = 0$.

Problem 2:

Expand $\frac{-1}{(z-1)(z-2)}$ as a power series in z in the regions

(i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$.

Solution:

Let $f(z) = \frac{-1}{(z-1)(z-2)}$. By splitting into partial fractions, we have

$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$.

(i) The only points where $f(z)$ is not analytic are 1 and 2. Hence $f(z)$ is analytic in $|z| < 1$ and hence can be represented as a Taylor's series in $|z| < 1$.

$$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$
$$= -\frac{1}{1-z} + \frac{1}{2-z}$$
$$= -(1-z)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$
$$= -(1 + z + z^2 + \cdots + z^n + \cdots)$$
$$+ \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots + \frac{z^n}{2^n} + \cdots\right)$$
$$= \sum_{n=0}^{\infty}\left[-z^n + \frac{1}{2}\left(\frac{z}{2}\right)^n\right]$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right)z^n$$

(ii) $f(z)$ is analytic in the annular region $1 < |z| < 2$ and hence can be expanded as a Laurent's series in this region.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$
$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$
$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$
$$= \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$
$$\left(\text{since } \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1\right)$$
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

This gives the Laurent's series expansion in $1 < |z| < 2$.

(iii) $f(z)$ is analytic in the domain $|z| > 2$ and in this domain we have $|2/z| < 1$. Hence

$$f(z) = \frac{1}{z}\left[\frac{1}{1-(1/z)}\right] - \frac{1}{z}\left[\frac{1}{1-(2/z)}\right]$$
$$= \frac{1}{z}[1-(1/z)]^{-1} - \frac{1}{z}[1-(2/z)]^{-1}$$
$$= \frac{1}{z}\left[\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right)-\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right)\right]$$
$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

Problem 3:

Expand $\frac{1}{z(z-1)}$ as Laurent's series (i) about $z = 0$ in powers of z and (ii) about $z = 1$ in powers $z - 1$. Also state the region of validity.

Solution:

(i) The only points where $f(z)$ is not analytic are 0 and 1. Hence $f(z)$ can be expanded as a Laurent's series in the annulus $0 < |z| < 1$.

$$f(z) = \frac{1}{z(z-1)}$$
$$= -\frac{1}{z}(1-z)^{-1}$$
$$= -\frac{1}{z}(1+z+z^2+\cdots+z^n+\cdots)(\text{ since } |z|<1)$$
$$= -\left(\frac{1}{z}+1+z+z^2+\cdots+z^n+\cdots\right)$$

This is the Laurent's series expansion of $f(z)$ in $0 < |z| < 1$.

ii) $f(z)$ is analytic in $0 < |z-1| < 1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$\frac{1}{z(z-1)} = \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$
$$= \frac{1}{z-1}[1+(z-1)]^{-1}$$
$$= \frac{1}{z-1}[1-(z-1)+(z-1)^2-(z-1)^3+\cdots]$$
$$= \frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots$$

This gives the Laurent's series expansion in $0 < |z-1| < 1$.

Problem 4:

Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution:

Let $f(z) = \frac{z}{(z+1)(z+2)}$

$$= \frac{-1}{z+1} + \frac{2}{z+2} \text{ (verify)}$$
$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$
$$= [1-(z+2)]^{-1} + \frac{2}{z+2}$$
$$= [1+(z+2)+(z+2)^2+\cdots] + \frac{2}{z+2}$$
$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

Problem 5:

Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent's series valid for (i) $|z| - 1$

(ii) $1 < |z| < 2$ (iii) $|z| > 2$ (iv) $|z-1| > 1$ and (v) $0 < |z-2| < 1$.

Solution:

Let $f(z) = \frac{z}{(z-1)(2-z)}$.

$\therefore f(z) = \frac{1}{z-1} + \frac{2}{2-z}$ (by partial fractions).

(i) $|z| < 1$.

$$f(z) = \frac{-1}{1-z} + \frac{2}{2(1-z/2)} = -(1-z)^{-1} + (1-z/2)^{-1}$$

Since $|z| < 1, f(z)$ can be expanded in series as

$$f(z) = -[1+z+z^2+z^3+\cdots] + \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$
$$= -\frac{z}{2} - \frac{3z^2}{4} - \frac{7z^3}{8} - \cdots$$

(ii) $1 < |z| < 2$.

$$f(z) = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)} = \frac{1}{z}(1-1/z)^{-1} + (1-z/2)^{-1}$$

Now $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$. Hence, we have

$$f(z) = \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$+\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$= \cdots + \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \frac{1}{z} + 1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots$$

(iii) $|z| > 2$. Hence $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$.

$$f(z) = \frac{1}{z(1 - 1/z)} - \frac{2}{z(1 - 2/z)}$$

$$= \frac{1}{z}(1 - 1/z)^{-1} - \frac{2}{z}(1 - 2/z)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) - \frac{2}{z}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 \cdots\right)$$

$$= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \cdots$$

(iv)$|z - 1| > 1$. Hence $\frac{1}{|z-1|} < 1$.

$$f(z) = \frac{1}{z - 1} - \frac{2}{z - 2}$$

$$= \frac{1}{z - 1} - \frac{2}{z - 1 - 1}$$

$$= \frac{1}{z - 1} - \frac{2}{(z - 1)\left(1 - \frac{1}{z - 1}\right)}$$

$$= \frac{1}{z - 1} - \frac{2}{z - 1}\left(1 - \frac{1}{z - 1}\right)^{-1}$$

$$= \frac{1}{z - 1} - \frac{2}{z - 1}\left[1 + \frac{1}{z - 1} + \left(\frac{1}{z - 1}\right)^2 + \left(\frac{1}{z - 1}\right)^3 + \cdots\right]$$

$$= -\frac{1}{z - 1} - \frac{2}{(z - 1)^2} - \frac{2}{(z - 1)^3} - \cdots$$

(v) $0 < |z - 2| < 1$.

$$f(z) = \frac{1}{z-2+1} - \frac{2}{z-2}$$

$$= [1 + (z-2)]^{-1} - \frac{2}{z-2}$$

$$= [1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots] - \frac{2}{z-2}$$

$$= \frac{-2}{z-2} + 1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots$$

Problem 6:

Expand $\frac{1}{z^2-3z+2}$ in Laurent's series valid in the region $1 < |z| < 2$.

Solution:

Let $f(z) = \frac{1}{z^2-3z+2}$. Then

$$f(z) = \frac{(z-2)-(z-1)}{(z-2)(z-1)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$f(z)$ is analytic in the region $1 < |z| < 2$.

Hence $f(z)$ can be expanded in Laurent's series in that region. Now

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}.$$

In the region $1 < |z| < 2$, we have $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$. Hence $f(z)$ can be expanded in Laurent's series as

$$f(z) = -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$-\frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Problem 7:

If $f(z)=\frac{z+4}{(z+3)(z-1)^2}$ find Laurent's series expansions in

(i) $0 \leq |z-1| < 4$ and (ii) $|z-1| > 4$.

Solution:

Let $f(z)=\frac{z+4}{(z+3)(z-1)^2}$.

By expressing $f(z)$ into partial fractions we get

$$f(z)=\frac{1}{16(z+3)}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}$$

(i) $0<|z-1|<4$. Hence $0<\left|\frac{z-1}{4}\right|<1$.

$$f(z)=\frac{1}{16(z-1+4)}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}$$

$$=\frac{1}{64\left(1+\frac{z-1}{4}\right)}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}$$

$$=\frac{1}{64}\left(1+\frac{z-1}{4}\right)^{-1}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}$$

Since $\left|\frac{z-1}{4}\right|<1$, we have

$$f(z)=\frac{1}{64}\left[1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+\cdots\right]$$

$$-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}$$

$$=\frac{5}{4(z-1)^2}-\frac{1}{16(z-1)}+\frac{1}{64}-\frac{1}{64}\left[\frac{z-1}{4}-\left(\frac{z-1}{4}\right)^2+\cdots\right]$$

This is the required Laurent's series expansion for $f(z)$ in $0<|z-1|<4$.

(ii) $|z-1|>4$. Hence $\left|\frac{4}{z-1}\right|<1$.

Now $f(z)=\frac{1}{16(z-1)\left(1+\frac{4}{z-1}\right)}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}$

$$= \frac{1}{16(z-1)}\left[1 - \left(\frac{4}{z-1}\right) + \left(\frac{4}{z-1}\right)^2 - \left(\frac{4}{z-1}\right)^3 + \cdots\right]$$

$$- \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \frac{4^2}{(z-1)^5} - \cdots$$

Problem 8:

Find the Laurent's series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ valid in the annular region $2 < |z| < 3$.

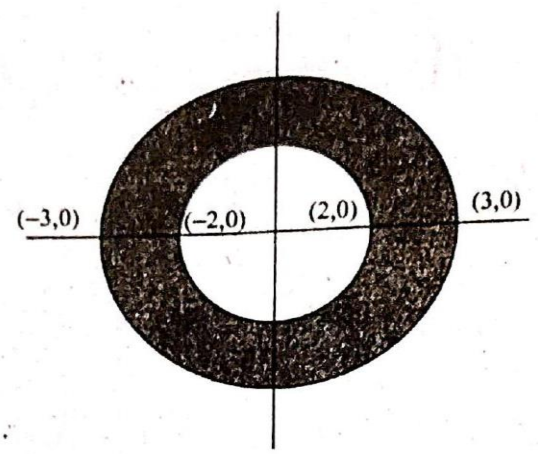
Solution:

Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$.

By splitting $f(z)$ into partial fractions, we get

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$f(z)$ is analytic in the annular region $2 < |z| < 3$.



Hence $f(z)$ can be expanded as a Laurent's series in that region.

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$-\frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

$$= 1 + 3\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^{n+1}}$$

Problem 9:

For the function $f(z) = \frac{2z^3+1}{z(z+1)}$ find (i) a Taylor's series valid in a neighbourhood of $z = i$ and (ii) a Laurent's series valid within an annulus of which centre is the origin.

Solution:

(i) $f(z) = \frac{2z^3+1}{z(z+1)}$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1} \text{ (by partial fractions)}$$

$$= 2(z-1) + \frac{1}{z} + \frac{1}{z+1} \quad \ldots\ldots\ldots\ldots(1)$$

$$= g(z) + h(z) + j(z)$$

where $g(z) = 2(z-1), h(z) = \frac{1}{z}$ and $j(z) = \frac{1}{z+1}$.

Taylor's expansion for $g(z)$ about $z = i$ is obviously $2(i-1) + 2(z-i)$.

Taylor's expansion for $h(z)$ about $z = i$ is given by

$$h(z) = h(i) + \sum_{n=1}^{\infty} \frac{h^{(n)}(i)}{n!}(z-i)^n$$

Here $h(i) = \frac{1}{i}$; $h^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$ so that $h^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$.

$h(z) = \frac{1}{i} + \sum_{n=1}^{\infty} \frac{(-1)^n n!}{i^{n+1} n!}(z-i)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}}(z-i)^n$.

Similarly we can prove that $j(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(1+i)^{n+1}}$.

Hence the Taylor's expansion for $f(z)$ is

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n$$

(ii) $f(z) = 2z - 2 + \frac{1}{z} + (1+z)^{-1}$ (from (1))

$$= 2z - 2 + \frac{1}{z} + (1 - z + z^2 - z^3 + \cdots) \text{ if } |z| < 1.$$

$\therefore$ In the annulus $0 < |z| < 1$ the Laurent's expansion is given by

$$f(z) = \frac{1}{z} - 1 + z + z^2 - z^3 + z^4 - \cdots$$

Problem 10:

Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$ as a Laurent's series. Also indicate the region of convergence of the series.

Solution:

$$f(z) = \frac{e^{2(z-1)+2}}{(z-1)^3}$$

$$= \frac{e^2 e^{2(z-1)}}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2^4}{4!}(z-1) + \cdots\right]$$

This series converges for all values of z except $z = 1$.

Exercises

1.Prove that $\frac{1+2z}{z^2+z^3} = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + z^3 - \cdots$ where $0 < |z| < 1$.

2.Find a Laurent's series expansions in powers of z of the function $f(z) = \frac{1}{z(1+z^2)}$.

3.Find two different Laurent's series for $\frac{1}{z^2(1-z)}$ about $z = 0$ and state the regions of validity.

4.Expand $\frac{1}{(z-1)(z-2)}$ as a power series in z valid in

(i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$

5.Expand $\frac{1}{z(z^2-3z+2)}$ as a power series valid in $1 < |z| < 2$.

6.Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid for

(i) $1 < |z| < 3$.

(ii) $|z| > 3$

(iii) $0 < |z+1| < 2$

(iv) $|z| < 1$

7.Expand in Laurent's series $\frac{1}{z(z-1)^2}$ at the point $z = 1$.

8.Find a Laurent's series expansion in powers of z for the function $f(z) = \frac{1}{z(1+z^2)}$.

9. Expand $\frac{1}{z^2(z-3)^2}$ as a Laurent's series at $z = 3$ and state the region of validity.

10.Expand $\frac{1}{z(z^2-3z+2)}$ in powers of z in the regions

(i) $0 < |z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

Answers.

2. $f(z) = \frac{1}{z} - z + z^3 - z^5 + \cdots; 0 < |z| < 1$

$$f(z) = \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} \cdots \cdots; |z| > 1.$$

3.(i) $\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots; 0 < |z| < 1$

(ii) $-\left(\frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots\right); |z| > 1$

4.(i) $\sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n$

(ii) $-\sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

(iii) $\sum_{n=1}^{\infty} \frac{2^n - 1}{z^{n+1}}$

5. $-\sum_{n=0}^{\infty} \left[\frac{1}{z^{n+2}} + \frac{z^{n-1}}{2^{n+1}}\right]$

6.(i) $-\frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \cdots$

(ii) $\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$

(iii) $\frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$

(iv) $\frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$

7. $\sum_{n=0}^{\infty} (-1)^n (z-1)^{n-2}$

8. $\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1}$ if $0 < |z| < 1$

9. $\frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots; 0 < |z-3| < 3$

10. (i) $\frac{1}{2z} + \frac{3}{4} + \frac{7z}{8} + \frac{15z^2}{16} + \frac{31z^3}{32} + \cdots$

(ii) $\left(-\frac{1}{2z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right) - \frac{1}{4}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots\right)$

(iii) $(2-1)\frac{1}{z^3} + (2^2-1)\frac{1}{z^4} + (2^3-1)\frac{1}{z^5} + \cdots$

4.6. Zeros of an Analytic Function:

Definition:

Let $f(z)$ be a function which is analytic in a region D. Let $a \in D$. Then a is said to be a zero of order r (where r is a positive integer) for $f(z)$ if $f(z) = (z - a)^r \varphi(z)$ where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Example 1:

Consider $f(z) = \sin z$

we know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\cdots$$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots\cdots\right.$$

$$= z\varphi(z)$$

where $\varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

Obviously $\varphi(z)$ is analytic and $\varphi(0) = 1 \neq 0$.

$z = 0$ is a zero of order 1 for $\sin z$.

Example 2:

Let $f(z) = (z - 2i)^2 (z + 3)^3 e^z$

$2i$ is a zero of order 2 and -3 is a zero of order 3 for $f(z)$.

Example 3:

Let $f(z) = z^2 \sin z$

$$\text{Then } f(z) = z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\right)$$

$$= z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots\right)$$

$$= z^3\varphi(z)$$

where $\varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ ....

Obviously $\varphi(z)$ is analytic and $\varphi(0) \neq 0$.

$\therefore\ z = 0$ is a zero of order 3 for $f(z) = z^2 \sin z$.

Example 4:

Let $f(z) = \frac{z^3-1}{z^3+1}$.

$f(z) = 0 \Rightarrow z^3 - 1 = 0$

$\Rightarrow (z-1)(z^2+z+1) = 0$

Hence the zeros of $f(z)$ are $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

and each one is a zero of order 1.

Theorem 1:

Suppose $f(z)$ is analytic in a region D and is not identically zero in D. Then the set of all zeros of $f(z)$ is isolated.

Proof:

Let $a \in D$ be a zero for $f(z)$. We shall prove that there exists a neighborhood $|z-a| < \delta$ such that this neighbourhood does not contain any other zero for $f(z)$. guppose a is a zero of order r for $f(z)$.

, then $f(z) = (z-a)^r \varphi(z)$ ... . (1)

where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Now, since φ is analytic at a, φ is continuous at a.

We can find a $\delta > 0$ such that

$$|z-a| < \delta \Rightarrow |\varphi(z) - \varphi(a)| < \frac{|\varphi(a)|}{2}$$

We claim that the neighborhood $|z-a| < \delta$ does not contain any other zero of $f(z)$. suppose $b \neq a$ is another zero for $f(z)$ in this neighbourhood. Then $|b - a| < \delta$ and $f(b) = 0$.

$\therefore\ (b-a)^r \varphi(b) = 0$. (from (1))

Now, since $b \neq a, (b-a)^r \neq 0$

$\therefore \varphi(b) = 0$

Further $|b-a| < \delta \Rightarrow |\varphi(b) - \varphi(a)| < \frac{|\varphi(a)|}{2}$

$\Rightarrow |\varphi(a)| < \frac{|\varphi(a)|}{2}$ which is a contradiction.

Thus the neighbourhood $|z-a| < \delta$ contains no other zero of $f(z)$ and hence the set of all zeros of $f(z)$ is isolated.

Corollary 1:

Let $f(z)$ be analytic in a region D. Suppose $f(z) = 0$ on a subset of D which has a limit point in D. Then $f(z)$ is identically zero in D.

Corollary 2:

Let $f(z)$ and $g(z)$ be two functions which are analytic in a region D. Suppose $f(z) = g(z)$ on a subset of D which has a limit point in D. Then $f(z) = g(z)$ in D.

(consider the function $f(z) - g(z)$ and the result follows from corollary 1)

Exercises.

1. Find all the zeros of the following functions.

   (a) $\cos z$ (b) $\frac{(z+1)^2(iz+2)^3}{z+7}$

2. Prove that there is no analytic functions whose zeros are precisely the points $1, \frac{1}{2}, \frac{1}{3}, , \cdots\cdots, \frac{1}{n}, \cdots$..

Answers.

1. (a) $(2n+1)\pi/2; n \in Z$ (b) -1 and $-2/i$.

4.7. Singularities:

Definition:

A point a is called a singular point or a singularity of a function $f(z)$ if $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z-a| < r$.

Example 1:

Consider the function $f(z) = \frac{1}{z}$.

Then $f'(z) = -\frac{1}{z^2}$ for all $z \neq 0$.

Thus $f(z)$ is analytic except at $z = 0$.

$\therefore z = 0$ is a singular point of $f(z)$.

Example 2:

Consider the function $f(z) = \frac{1}{z(z-i)}$.

0 and i are singular points for $f(z)$.

Definition:

A point a is called an isolated singularity for $f(z)$ if

(i) $f(z)$ is not analytic at $z = a$ and

(ii) there exists $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$.

(i.e) the neighbourhood $|z - a| < r$ contains no singularity of $f(z)$ except a.

Example 1:

$f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities $z = 0, i, -i$.

Example 2:

Consider the principal branch of logarithm given by $\log re^{i\theta} = \log r + i\theta$ where $-\pi < \theta \leq \pi$.

All points on the negative real axis are singular points of this function. These singularities are not isolated.

Example 3:

Consider the function $f(z) = \frac{1}{\sin z}$. The singular points are $0, \pm\pi, \pm 2\pi, \ldots \ldots$ and these are isolated singular points.

We now proceed to classify the isolated singularities of a function.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z - a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}} \text{ and } b_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - a)^{-n+1}}$$

The series consisting of the negative powers of $z-a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the principal part or singular and of $f(z)$ at $z=a$.

The singular part of $f(z)$ at $z=a$ determines the character of the singularity. there are three types of singularities. They are

(i) **Removable singularities**

(ii) **Poles**

(iii) **Essential singularities.**

Definition:

Let a be an isolated singularity for $f(z)$. Then a is called a removable singularity if the principal part of $f(z)$ at $z=a$ has no terms.

Note. If a is a removable singularity for $f(z)$ then the Laurent's series expansion of $f(z)$ about $z=a$ is given by

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$$
$$=a_0+a_1(z-a)+\cdots+a_n(z-a)^n+\cdots$$

Hence $\lim_{z\to a} f(z)=a_0$

Hence by defining $f(a)=a_0$ the function $f(z)$ becomes analytic at a.

Example 1:

Let $f(z)=\frac{\sin z}{z}$. Clearly 0 is an isolated singular point for $f(z)$.

Now, $$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)$$
$$=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots$$

Here the principal part of $f(z)$ at $z=0$ has no terms.

Hence $z=0$ is a removable singularity.

Also $\lim_{z\to 0} \frac{\sin z}{z} = 1$. Hence the singularity can be removed by defining $f(0) = 1$ so that the extended function becomes analytic at $z = 0$.

Example 2:

Let $f(z) = \frac{z-\sin z}{z^3}$.

$z = 0$ is an isolated singularity.

Further
$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right]$$
$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots\cdots$$

$\therefore z = 0$ is a removable singularity.

By defining $f(0) = \frac{1}{6}$ the function becomes analytic at $z = 0$.

Definition:

Let a be an isolated singularity of $f(z)$. The point a is called a pole if the principal part of $f(z)$ at $z = a$ has a finite number of terms. If the principal part of $f(z)$ at $z = a$ is given by

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots\cdots + \frac{b_r}{(z-a)^r}$$

where $b_r \neq 0$, we say that a is a pole of order r for $f(z)$.

Vote. A pole of order 1 is called a simple pole and a pole of order 2 is called a double pole.

Example 1:

Consider $f(z) = \frac{e^z}{z}$.

$$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\cdots$$

Here the principal part of $f(z)$ at $z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$.

Example 2:

Let $f(z) = \tan z = \frac{\sin z}{\cos z}$. The singularities of $f(z)$ are $\frac{\pi}{2} + n\pi$, where $n \in \mathbf{Z}$. All the singularities are poles of order 1.

Example 3. $f(z) = \frac{\cos z}{z^2}$ has a double pole at $z = 0$.

For, $$\begin{aligned}\frac{\cos z}{z^2} &= \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\cdots\right)\\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \cdots\cdots\end{aligned}$$

Example 4:

Let $f(z) = \frac{z^2-2z+3}{z-2}$

then $f(z) = 2 + (z-2) + \frac{3}{z-2}$ (by partial fractions)

Here $f(z)$ has a simple pole at $z = 2$.

Definition:

Let a be an isolated singularity of $f(z)$. The point a is called an essential apularity of $f(z)$ at $z = a$ if the principal part of $f(z)$ at $z = a$ has an infinite number geems.

Example 1:

Let $f(z) = e^{1/z}$. Obviously $z = 0$ is an isolated singularity for $f(z)$. Further $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$. The principal part of $f(z)$ has infinite number of terms. Hence $e^{1/z}$ has an essential singularity at $z = 0$.

Example 2:

Let $f(z) = z^2\sin(1/z)$. $f(z)$ has essential singularity at $z = 0$.

In the following theorem we give equivalent characterisations for an isolated singular wint a of $f(z)$ to be a removable singularity.

Theorem 1:

Let $f(z)$ be a function defined in a region D of the complex plane except possibly at a point $a \in D$ and let a be an isolated singularity for $f(z)$. Then a is a removable singularity for $f(z)$ if and only if there exists a complex number a_0 such that by defining $f(a) = a_0$ the extended function becomes analytic at a.

Proof:

Suppose a is a removable singularity for $f(z)$.

$$\text{Then } f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, 0 < |z-a| < r$$
$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots$$

$\therefore$ By defining $f(z) = a_0, f$ becomes analytic at a. Conversely, suppose there exists a complex number a_0 such that by defining $f(a) = a_0, f$ becomes analytic in $|z-a| < r$.

Hence f can be represented as a Taylor's series, in power of $z-a$ in this neighbourhood, given by $f(z) = \sum_0^\infty a_n(z-a)^n$. This shows that the principal part of $f(z)$ at $z = a$ has no terms. Hence a is a removable singularity for $f(z)$.

Theorem 2: (Riemann's theorem)

Let f be a function which is bounded and analytic throughout a domain $0 < |z-z_0| < \delta$. Then either f is analytic at z_0 or else z_0 is a removable singular point of f.

Proof:

Consider the Laurent's series for the function in the given domain about z_0. The coefficient b_n of $\frac{1}{(z-z_0)^n}$ is given by $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$ where C is the circle $|z-z_0| = r$ where $r < \delta$.

Now, since f is bounded there exists a positive real number M such that $|f(z)| \leq M$ in $0 < |z-z_0| < \delta$.

$$\therefore\ |b_n| \leq \frac{1}{2\pi}\frac{M(2\pi r)}{r^{-n+1}}$$
$$= Mr^n$$

Since it is true for every r such that $0 < r < \delta$, taking limit $r \to 0$ we get $b_n = 0$.

Hence the Laurent's series for $f(z)$ has no principal part. Hence the theorem follows.

Theorem 3:

Let $f(z)$ be a function having a as an isolated singular point. The the following are equivalent.

(i) a is a pole of order r for $f(z)$.

(ii) $f(z)$ can be written in the form $f(z) = \frac{1}{(z-a)^r}\theta(z)$ where $\theta(z)$ has a removable singularity at $z = a$ and $\lim_{z\to a}\theta(z) \neq 0$.

(iii) a is a zero of order r for $\frac{1}{f(z)}$.

Proof:

(i) $\Rightarrow$ (ii). Let a be a pole order r for $f(z)$. Then the Laurent's series expansion of $f(z)$ about a is given by $f(z) = \sum_{n=1}^{r} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n(z-a)^n$ where $b_r \neq 0$.

$$\therefore\ f(z) = \frac{1}{(z-a)^r}[b_r + b_{r-1}(z-a) + \cdots + b_0(z-a)^{r-1} + a_0(z-a)^r + \cdots]$$

$$= \frac{1}{(z-a)^r}\dot{\theta}(z) \text{ where } \theta(z) = b_r + b_{r-1}(z-a) + \cdots \ldots$$

Clearly $\lim_{z\to a}\theta(z) = b_r \neq 0$ and $\theta(z)$ has a removable singularity at $z = a$.

(ii) $\Rightarrow$ (iii) Let $f(z) = \frac{1}{(z-a)^r}\theta(z)$ and by suitably defining $\theta(a)$ we may assume that $\theta(z)$ is analytic at a and $\theta(a) \neq 0$.

$$\therefore\ \frac{1}{f(z)} = (z-a)^r \frac{1}{\theta(z)} \text{ and } \frac{1}{\theta(z)} \text{ is analytic at } a \text{ and } \frac{1}{\theta(a)} \neq 0.$$

Hence a is a zero of order r for $\frac{1}{f(z)}$.

(if) $\Rightarrow$ (i) Let a be a zero of order r for $\frac{1}{f(z)}$, Then $\frac{1}{f(z)} = (z-a)^r g(z)$ where $g(z)$ is analytic at a and $g(a) \neq 0$.

$\therefore f(z) = \frac{g_1(z)}{(z-a)^r}$ where $g_1(z)$ is analytic at a and $g_1(a) \neq 0$.

Let $g_1(z) = a_0 + a_1(z-a) + \cdots + a_n(z-a)^n + \cdots$.. so that $a_0 \neq 0$.

$$\therefore \; f(z) = \frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \cdots + a_r + a_{r+1}(z-a) + \cdots$$

$$\text{in } 0 < |z-a| < r$$

The principal part of $f(z)$ at $z = a$ is

$$\frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \cdots\cdots + \frac{a_{r-1}}{z-a} \text{ and } a_0 \neq 0.$$

$\therefore$ a is a pole of order r for $f(z)$.

Theorem 4:

An isolated singularity a of $f(z)$ is a pole if and only if $\lim_{z \to a} f(z) = \infty$.

proof. If a is a pole of order r for $f(z)$ then $f(z) = \frac{g(z)}{(z-a)^r}$ with $g(a) \neq 0$.

$\therefore \lim_{z \to a} f(z) = \infty$.

Conversely let a be an isolated singularity for $f(z)$ and let $\lim_{z \to a} f(z) = \infty$.

Let $\theta(z) = \dfrac{1}{f(z)}$

Then $\lim_{z \to a} \theta(z) = 0$

Hence a is a removable singularity for $\theta(z)$ and by defining $\theta'(z) = 0, \theta$ becomes analytic at a. Let a be a zero of order r for the function $\theta(z)$. Then a is a pole of order r for $f(z)$.

Definition:

A function $f(z)$ is said to be a meromorphic function if it is analytic except at a finite number of points and these finite set of points are poles.

Example 1: Let $f(z) = \frac{1}{z(z-1)^2}$.

$f(z)$ is analytic except at $z = 0$ and $z = 1$. Also 0 and 1 are poles of order 1 and 2 respectively. Hence $f(z)$ is a meromorphic function.

Example 2:

$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$ is a meromorphic function.

Example 3:

$e^{1/z}$ is not a meromorphic function since $z = 0$ is an essential singularity for $e^{1/z}$.

The following theorem due to Weierstrass describes the behavior of a function in the neighbourhood of an essential singularity.

Theorem 5:

Let z_0 be an essential singularity for a function $f(z)$. Let c be any complex number. Then given $\varepsilon, \delta > 0$ there exists a point z such that $|z - z0| < \delta$ and $|f(z) - c| < \varepsilon$.

(i.e) The function $f(z)$ comes arbitrarily close to any complex number c in every neighbourhood of an essential singularity.

Proof:

Suppose the theorem is false. Then there exist $\delta, \varepsilon > 0$ such that for every point z satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - c| \geq \varepsilon$.

Now consider the function $g(z) = \frac{1}{f(z)-c}$.

$$\therefore |g(z)| = \frac{1}{|f(z) - c|} \leq \frac{1}{\varepsilon}$$

Hence $g(z)$ is bounded and further $g(z)$ is analytic in $0 < |z - z_0| < \delta$.

Hence by Riemann's theorem $z = z_0$ is a removable singularity for $g(z)$.

Now, if $g(z_0) \neq 0$ then $\frac{1}{g(z)} = f(z) - c$ is analytic at z_0.

$\therefore$ By suitably defining $g(z_0)$, the function $g(z)$ becomes analytic at z_0.

If $g(z_0) = 0$ then let z_0 be a zero of order r for $g(z)$.

Then z_0 is a pole of order r for $\frac{1}{g(z)} = f(z) - c$.

Thus $f(z)$ is either analytic at z_0 or else z_0 is a pole of $f(z)$ which is a contradiction to the hypothesis that z_0 is an essential singularity for $f(z)$. Hence the theorem.

Solved problems

Problem 1:

Determine and classify the singular points of $f(z) = \frac{z}{e^z-1}$.

Solution. The singularities of $f(z)$ are given by the values of z for which $e^z - 1 = 0$.

Hence $z = 2n\pi i, n \in \mathbf{Z}$, are the singularities of $f(z)$.

$$\text{Now,} e^z - 1 = \left(1 + z + \frac{z^2}{2!} + \cdots\cdots + \frac{z^n}{n!} + \cdots\right) - 1.$$

$$= z + \frac{z^2}{2!} + \cdots\cdots + \frac{z^n}{n!} + \cdots\cdots$$

$\lim_{z\to 0} \dfrac{z}{e^z - 1} = 1$

Hence 0 is a removable singularity for $f(z)$.

Also, $\lim_{z\to 2n\pi i}\left(\frac{z}{e^z-1}\right) = \infty$ if $n \neq 0$ and hence $2n\pi i, n \neq 0$, are simple poles of $f(z)$.

Problem 2:

Determine and classify the singularities of $f(z) = \sin(1/z)$.

Solution:

Clearly 0 is the only singularity of $f(z)$.

Also $f(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots\cdots$.

Thus the principal part of $f(z)$ at $z = 0$ has infinitely many terms and hence 0 is an essential singularity for $f(z)$.

Problem 3: Determine and classify the singular points of $\frac{1}{(2\sin z-1)^2}$.

Solution:

The singularities of $f(z)$ are given by the values of z for which $2\sin z - 1 = 0$.

$\therefore$ The singularities of $f(z)$ are given by $z = \frac{\pi}{6} + 2n\pi, n \in \mathbf{Z}$, and they are double poles.

Exercises.

1.Find the singularities of the following functions and classify the singularities.

(i) $ze^{1/z}$ (ii) $\frac{z^2}{1+z}$ (iii) $\frac{\sin z}{z}$ (iv) $\frac{e^{1/z}}{z-1}$ (v) $\frac{z}{e^{1/z}-1}$

(vi) $\sin\left(\frac{1}{1-z}\right)$ (vii) $\frac{z^2-2z+3}{z-2}$ (viii) $(z-i)\sin\left(\frac{1}{z+2i}\right)$

2.Show that the singular points of each of the following functions are poles. Determine the order of each pole.

(i) $\frac{z+1}{z^2-2z}$ (ii) $\tanh z$ (iii) $\frac{1-e^{2z}}{z^4}$ (iv) $\frac{e^{2z}}{(z-1)^2}$

(v) $\frac{1}{z^2+1}$ (vi) $\frac{1}{z^2(z-3)^2}$ (vii) $\frac{1}{z^4+2z^2+1}$ (viii) $\frac{2(1+2)}{1-\cos 2}$

(ix) $\frac{z^2-2z+3}{z-2}$ (x) $(z-i)\sin\left(\frac{1}{z+2i}\right)$

3.Find the order of the pole $z = 0$ for the following functions.

(i) $\frac{e^2}{z}$ (ii) $\frac{e^2}{z^2}$ (iii) $\frac{1-\sin z}{z^5}$

4.Let f and g have a pole of order m and n respectively at a. What can be said about the order of pole of (i) $f+g$

(ii) fg

(iii) f/g at a

5.Show that if f has an essential singularity at a so does f^2

Answers

2. (i) 0 and 2 are simple poles (ii) 0 is a simple pole (iii) 0 is a pole of order 3. (iv) 1 is a double pole (v) i and $-i$ are simple poles (vi) 0,3 are double poles (vii) $i, -i$ are double poles (viii) 0 is a simple pole. (ix) 2 is a pole (x) $-2i$ is an essential singularity.

3. (i) 1 (ii) 2 (iii) 5 .

UNIT V

Residues–Cauchy Residue theorem–Residue at infinity–Evaluation of Definite Integrals.

Chapter 5: Sections 5.1 to 5.3

5.Calculus of Residues

Introduction

this chapter we introduce the concept of the residue of a function $f(z)$ at an isolated angular point and prove Cauchy's residue theorem. Using this theorem, we evaluate certain types of real definite integrals.

5.1. Residues:

Definition:

Let a be an isolated singularity for $f(z)$. Then the residue of $f(z)$ at a is defined to be the coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ about a and is denoted by Res$\{f(z);a\}$.

Thus Res $\{f(z);a\} = \frac{1}{2\pi i}\int_C f(z)dz = b_1$ where C is a circle $|z-a| = r$ such that f is analytic in $0. < |z-a| < r$.

Example:

Consider $f(z) = \frac{e^z}{z^2}$

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

$\therefore f(z)$ has a double pole at $z = 0$.

$\therefore$ Res$\{f(z);0\} =$ coefficient of $\frac{1}{z} = 1$.

The following lemmas provide methods for calculation of residues.

Lemma 1:

If $z = a$ is a simple pole for $f(z)$ then $\text{Res}\{f(z); a\} = \lim_{z\to a}(z-a)f(z)$

Proof:

Since $z = a$ is a simple pole for $f(z)$ the Laurent's series expansion for $f(z)$ about $z = a$ is given by $f(z) = \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots\cdots$

Now, $(z-a)f(z) = b_1 + a_0(z-a) + a_1(z-a)^2 + \cdots$

$\lim_{z\to a}(z-a)f(z) = b_1 = \text{Res}\{f(z); a\}$

Lemma 2:

If a is a simple pole for $f(z)$ and $f(z) = \frac{g(z)}{z-a}$ where $g(z)$ is analytic at a and $g(a) \neq 0$ then $\text{Res}(f(z); a) = g(a)$.

Proof. By Lemma 1, Res $(f(z); a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a} g(z) = g(a)$.

Lemma 3:

If a is a simple pole for $f(z)$ and if $f(z)$ is of the form $\frac{h(z)}{k(z)}$ where $h(z)$ and $k(z)$ are analytic at a and $h(a) \neq 0$ and $k(a) = 0$ then $\text{Res}\{f(z); a\} = \frac{h(a)}{k'(a)}$

Proof:

Res $\{f(z); a\} = \lim_{z\to a}(z-a)f(z)$

$$= \lim_{z\to a}(z-a)\frac{h(z)}{k(z)}$$

$$= \lim_{z\to a} h(z) \lim_{z\to a} \frac{(z-a)}{k(z)}.$$

$$= \lim_{z\to a} h(z) \lim_{z\to a}\left[\frac{z-a}{k(z)-k(a)}\right] \ (\text{ since } k(a) = 0)$$

$$= h(a)\left[\frac{1}{k'(a)}\right] \frac{1}{z(z)-k(a)}$$

$$= \frac{h(a)}{k'(a)}.$$

Lemma 4:

Let a be a pole or order $m > 1$ for $f(z)$ and let $f(z) = \frac{g(z)}{(z-a)^m}$ where $g(z)$ is analytic at a and $g(a) \neq 0$. Then $\text{Res}\{f(z); a\} = \frac{g^{(m-1)}(a)}{(m-1)!}$

Proof:

$g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i} \int_C \frac{g(z)dz}{(z-a)^m}$ (by theorem on higher derivatives) where ✓ is a circle $|z-a| = r$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

$$\therefore \frac{g^{(m-1)}(a)}{(m-1)!} = \frac{1}{2\pi i} \int_C f(z)dz = \text{Res}\{f(z); a\}$$

Solved Problems

Problem 1:

Calculate the residue of $\frac{z+1}{z^2-2z}$ at its poles.

Solution:

Let $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$.

$z = 0$ and $z = 2$ are simple poles for $f(z)$.

$$\text{Res}\{f(z); 0\} = \lim_{z \to 0} (z-0) \left[\frac{z+1}{z(z-2)}\right]$$
$$= \lim_{z \to 0} \frac{z+1}{z-2} = -\frac{1}{2}$$
$$\text{Res}\{f(z); 2\} = \lim_{z \to 2} (z-2) \left[\frac{z+1}{z(z-2)}\right]$$
$$= \lim_{z \to 2} \frac{z+1}{z} = \frac{3}{2}$$

Aliter. $f(z)$ can be written as $f(z) = \frac{h(z)}{k(z)}$ where $h(z) = z+1$ and $k(z) = z^2 - 2z$ so that $k'(z) = 2z - 2$.

$$\therefore \ \text{Res}\{f(z);0\} = \frac{h(0)}{k'(0)} (\text{ by Lemma 3})$$
$$= -\frac{1}{2}$$
$$\text{Res}\{f(z);2\} = \frac{h(2)}{k'(2)} = \frac{3}{2}$$

Problem 2:

Find the residue at $z = 0$ of $\frac{1+e^z}{z\cos z+\sin z}$.

Solution:

Let $f(z) = \frac{1+e^z}{z\cos z+\sin z}$.

Clearly 0 is a pole of order 1 for $f(z)$.

$\therefore$ $\text{Res}\{f(z);0\} = \lim_{z\to 0} \frac{h(z)}{k'(z)}$ where $h(z) = 1 + e^z$ and $k(z) = z\cos z + \sin z$.

Now $k'(z) = -z\sin z + \cos z + \cos z = -z\sin z + 2\cos z$.

$$\therefore \ \text{Res}\{f(z);0\} = \frac{2}{2} = 1$$

Problem 3:

Find the residue of $\frac{1}{(z^2+a^2)^2}$ at $z = ai$.

Solution:

Let $f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ai)^2(z-ai)^2}$.

$z = ai$ and $z = -ai$ are poles of order 2 for $f(z)$.

$$\text{Let} g(z) = \frac{1}{(z+ai)^2}.$$
$$\therefore g'(z) = \frac{-2}{(z+ai)^3}.$$
$$\therefore \ \text{Res}\{f(z);ai\} = g'(ai) = \frac{-2}{(ai+ai)^3} = \frac{-2}{8a^3i^3} = \frac{2}{8a^3i}$$
$$= \frac{-i}{4a^3}$$

Problem 4:

Find the poles of $f(z)=\frac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

Solution:

$$f(z)=\frac{z^2+4}{z^3+2z^2+2z}=\frac{z^2+4}{z(z+1-i)(z+1+i)}.$$

$\therefore$ $0, i-1$ and $-1-i$ are simple poles for $f(z)$.

Here $f(z)=\frac{h(z)}{k(z)}$ where $h(z)=z^2+4$ and $k(z)=z^3+2z^2+2z$.

Hence $k'(z)=3z^2+4z+2$.

$$\text{Res}\{f(z);0\}=\frac{h(0)}{k'(0)}=\frac{4}{2}=2$$

$$\begin{aligned}\text{Res}\{f(z);i-1\}&=\frac{h(i-1)}{k'(i-1)}\\&=\frac{(i-1)^2+4}{3(i-1)^2+4(i-1)+2}\\&=\frac{3i-1}{2}\text{(after simplification)}\end{aligned}$$

Similarly,$\text{Res}\{f(z);-1-i\}=\frac{-(1+3i)}{2}$.

Problem 5:

Find the residue of $\cot z$ at $z=0$.

Solution:

$z=0$ is a simple pole for $\cot z$. Let $f(z)=\frac{\cos z}{\sin z}=\frac{h(z)}{k(z)}$.

$$\therefore\ \text{Res}\{f(z);0\}=\frac{h(0)}{k'(0)}=\frac{\cos 0}{\cos 0}=1$$

Problem 6:

Find the residue of $\frac{e^z}{z^2(z^2+9)}$ at its poles.

Solution: Let $f(z)=\frac{e^z}{z^2(z^2+9)}$.

Here $z=0$ is a double pole and $z=3i$ and $z=-3i$ are simple poles for $f(z)$.

To find the Res $\{f(z);0\}$, let $g(z) = \frac{e^z}{z^2+9}$.

Clearly $g(z)$ is analytic at $z = 0$ and $g(0) \neq 0$.

$$\text{Also } g'(z) = e^z\left[\frac{(z^2+9)-2z}{(z^2+9)^2}\right]$$

$$\therefore \text{ Res}\{f(z);0\} = \frac{g'(0)}{1!} \text{(by Lemma 4)}$$

$$= \frac{1}{9}$$

Now, to find Res $\{f(z);3i\}$; let $f(z) = \frac{h(z)}{k(z)}$ so that $h(z) = e^z$ and $k(z) = z^2(z^2+9)$.

Then $k'(z) = 4z^3 + 18z$

$$\therefore \text{ Res}\{f(z);3i\} = \frac{h(3i)}{k'(3i)}$$

$$= \frac{e^{3i}}{4(3i)^3 + 18(3i)}$$

$$= \frac{e^{3i}}{-108i + 54i}$$

$$= -\frac{e^{3i}}{54i}$$

$$= \frac{i(\cos 3 + i\sin 3)}{54}$$

Similarly, Res $(f(z);-3i) = -\frac{(\sin 3 + i\cos 3)}{54}$.

Problem 7:

Use Laurent's series to find the residue of $\frac{e^{2z}}{(z-1)^2}$ at $z = 1$.

Solution:

Let $f(z) = \frac{e^{2z}}{(z-1)^2}$.

First we expand $f(z)$ as Laurent's series at $z = 1$.

$$f(z)=\frac{e^{2(z-1)+2}}{(z-1)^2}$$
$$=\frac{e^2e^{2(z-1)}}{(z-1)^2}$$
$$=\frac{e^2}{(z-1)^2}\left[1+\frac{2(z-1)}{1!}+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+\cdots\right]$$
$$=e^2\left[\frac{1}{(z-1)^2}+\frac{2}{z-1}+2+\frac{4}{3}(z-1)+\cdots\right]$$

This is Laurent's series expansion for $f(z)$ at $z=1$.

$\operatorname{Res}\{f(z);1\}=$ coefficient of $\dfrac{1}{z-1}$ in Laurent's expansion

$=2e^2$

Note. Without expanding in Laurent's series the residue at $z=1$ can be found as follows. Since $f(z)$ has a pole of order 2 at $z=1$ we choose $g(z)=e^{2z}$.

$$\therefore\ \operatorname{Res}\{f(z);1\}=\frac{g'(1)}{1!}=\left[\frac{2e^{2z}}{1!}\right]_{z=1}=2e^2$$

Problem 8:

Find the residue of $\frac{ze^z}{(z-1)^3}$ at its pole.

Solution. Let $f(z)=\frac{ze^z}{(z-1)^3}$.

$z=1$ is a pole of order 3 for $f(z)$.

Let $g(z)=ze^z$ so that $g'(z)=e^z(z+1)$ and $g''(z)=e^z(z+2)$.

Then Res $\{f(z);1\}=\frac{g''(1)}{2!}=\frac{3e}{2}$.

Problem 9:

Find the residue of $\frac{1}{z-\sin z}$ at its pole.

Solution:

Let $f(z)=\frac{1}{z-\sin z}$.

$$\text{Now } z - \sin z = z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots$$

$$= z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \cdots\right)$$

z= 0 is a pole of order 3 for $f(z)$ and $f(z) = \frac{1}{z^3\left(\frac{1}{3!}-\frac{z^2}{5!}+\cdots\right)}$.

Now let $g(z) = \frac{1}{\left(\frac{1}{3!}-\frac{z^2}{5!}+\cdots\right)}$.

Then Res $\{f(z); 0\} = \frac{g''(0)}{2!}$. Clearly $g(0) = 6$.

Now $\frac{1}{g(z)} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$. Differentiating with respect to z, we have

$$-\frac{g'(z)}{[g(z)]^2} = -\frac{2z}{5!} + \frac{4z^3}{7!} - \cdots$$

Hence $g'(0) \doteq 0$.

Again differentiating with respect to z we have

$$\frac{[g(z)]^2[-g''(z)] + g'(z)2g(z)g'(z)}{[g(z)]^4} = \frac{-2}{5!} + \frac{12z^2}{7!} - \cdots$$

Putting $z = 0$ and using $g(0) = 6$ and $g'(0) = 0$ we get $\frac{-g''(0)}{36} = \frac{-2}{5!}$.

Hence $g''(0) = \frac{3}{5}$.

$\therefore$ Res$\{f(z); 0\} = \frac{g''(0)}{2!} = \frac{3}{10}$.

Exercises

1. Find the order of each pole and find the residue at the poles for each of the following functions.

(i) $\frac{z}{z^2+1}$ (ii) $\frac{z+1}{z^2-2z}$ (iii) $\frac{2z+3}{z(z^2+1)}$ (iv) $\frac{z^2}{z^2+a^2}$

(v) $\frac{1}{z^4+2z^2+1}$ (vi) $\frac{1}{z^2e^z}$ (vii) $\frac{1}{z^3(z+4)}$ (viii) $\frac{z+1}{z^2(z-2)}$

(ix) $\frac{\cos z}{z^3}$ (x) $\frac{\sin z}{z^4}$ (xi) $\left(\frac{z+1}{z-1}\right)^2$ (xii) $\frac{e^{2z}}{(z-1)^2}$

(xiii) $\frac{e^z}{z^2+\pi^2}$ (xiv) $\frac{1-e^{2z}}{z^4}$ (xv) $\frac{e^{iz}}{z^2+a^2}$ (a real) (xvi) $\frac{2z}{(z+4)(z-1)^2}$

2. Find the residue of $\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles.

3. Find the residue of $\frac{1+e^z}{\sin z+z\cos z}$ at the pole $z=0$.

4. Calculate the residue of $\sec^2 z$ at $z=\frac{\pi}{2}$.

   (Hint: $\sec^2 z = \frac{1}{\cos^2 z} = \frac{2}{1+\cos 2z}$).

5. Prove that the residue of $\frac{z^{2n}}{(1+z)^n}$ where $n \in \mathbf{N}$ at $z=-1$ is $\frac{(-1)^{n+1}(2n)!}{(n-1)!(n+1)!}$.

6. Find the residue of $\frac{1}{(1+z^2)^n}$ at $z=i$.

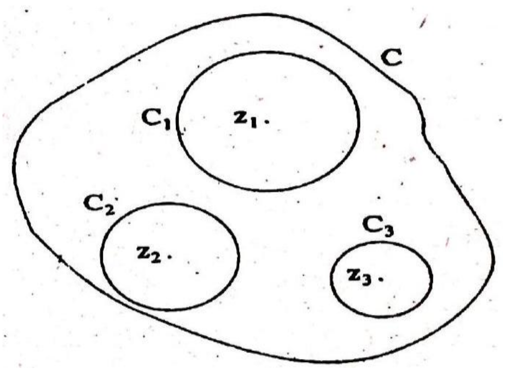
7. Prove that (i) $\text{Res}\{\tan z, \pi/2\} = -1$; (ii) $\text{Res}\left\{\frac{1-\cos z}{z^3}, 0\right\} = \frac{1}{2}$.

8. Show that all the singular points of $\frac{1}{z(e^z-1)}$ are poles. Find the order of poles and find the radius at the poles.

5.2. Cauchy's Residue Theorem

Theorem 1: (Cauchy's residue theorem)

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, \ldots\ldots, z_n$ inside C.



Then $\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_j\}$

Proof:

Let $C_1, C_2, \cdots, C_n$ be circles with centres $z_1, z_2, \cdots\cdots, z_n$ respectively such that all circles are interior to C and are disjoint with each other. (refer figure).

By Cauchy's theorem for multiply connected regions, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

$$= 2\pi i \text{Res}\{f(z); z_1\} + 2\pi i \text{Res}\{f(z); z_2\} + \cdots + 2\pi i \text{Res}\{f(z); z_n\}$$

(by definition of residue)

$$= 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_j\}.$$ Hence the theorem.

Example:

Evaluate $\int_C \frac{z^2 dz}{(z-2)(z+3)}$ where C is the circle $|z| = 4$.

Let $f(z) = \dfrac{z^2}{(z-2)(z+3)}$

$z = 2$ and $z = -3$ are simple poles for $f(z)$ and both of them lie inside $|z| = 4$.

Now, Res $\{f(z); 2\} = \lim_{z-2} (z-2)\left[\frac{z^2}{(z-2)(z+3)}\right] = \frac{4}{5}$.

$\text{Res}\{f(z); -3\} = \lim_{z\to -3} (z+3)\left[\frac{z^2}{(z-2)(z+3)}\right] = -\frac{9}{5}$.

$\therefore$ By Residue theorem $\int_C f(z)dz = 2\pi i\left[\frac{4}{5} + \left(-\frac{9}{5}\right)\right]$

$$= -2\pi i$$

$$\int_C \frac{z^2 dz}{(z-2)(z+3)} = -2\pi i$$

Theorem 2: (Argument theorem)

Let f be a function which is analytic inside and on a simple closed curve C except for a finite number of poles inside C. Also let $f(z)$ have no zeros on C. Then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P$ where N is the number of zeros of $f(z)$ inside C and P is the number of poles of $f(z)$ inside C. (A pole or zero of order m is counted m times).

Solution:

We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of $f(z)$ lying inside C.

Let z_0 be a zero of order n for $f(z)$. Let C_1 be a circle with centre z_0 such that it is the only zero of $f(z)$ inside C_1.

Then $f(z) = (z - z_0)^n g(z)$ where $g(z)$ is analytic and nonzero inside C_1.

Hence $f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$.

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)} \ldots\ldots\ldots..(1)$$

Since $g(z)$is analytic and non zero inside C_1, $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a Taylor's series about z_0.

$$\therefore \text{ Res}\left\{\frac{f'(z)}{f(z)}; z_0\right\} = \text{ coefficient of } \frac{1}{z - z_0} \text{ in } (1)$$

$$= n$$

Similarly if z_1 is a pole of order p for $f(z)$, then Res $\left\{\frac{f'(z)}{f(z)}; z_1\right\} = -p$.

Hence by Cauchy's residue theorem, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P$ where N is the number of zeros and P is the number of poles of $f(z)$ within C.

Corollary.

If $f(z)$ is analytic inside and on C and not zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N$ where N is the number of zeros lying inside C.

proof. Since the number of poles is zero, we have $P = 0$.

Hence the result follows.

Theorem 3. (Rouche's theorem)

If $f(z)$ and $g(z)$ are analytic ^{i}, side and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Proof. $f(z) + g(z) = f(z)\left[1 + \frac{g(z)}{f(z)}\right] = f(z)\varphi(z)$ where $\varphi(z) = 1 + \frac{g(z)}{f(z)}$.

Hence $[f(z) + g(z)]' = f'(z) + g'(z) = f'(z)\varphi(z) + f(z)\varphi'(z)$.

$$\therefore \frac{f'(z) + g'(z)}{f(z) + g(z)} = \frac{f'(z)\varphi(z) + f(z)\varphi'(z)}{f(z)\varphi(z)} = \frac{f'(z)}{f(z)} + \frac{\varphi'(z)}{\varphi(z)}$$

$$\therefore \frac{1}{2\pi i}\int_C \left[\frac{f'(z) + g'(z)}{f(z) + g(z)}\right] dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i}\int_C \frac{\varphi'(z)}{\varphi(z)} dz \ldots\ldots..(1)$$

Now, by hypothesis $|g(z)| < |f(z)|$ and hence $\left|\frac{g(z)}{f(z)}\right| < 1$ on C.

$$\therefore |\varphi(z) - 1| - 1 \text{ on C}$$

Hence by maximum modulus theorem, $|\varphi(z) - 1| < 1$ for every point z inside C.

$\therefore \varphi(z) \neq 0$ for every point inside C.

$$\text{Hence } \int_C \frac{\varphi'(z)}{\varphi(z)} dz = \text{ Number of zeros of } \varphi(z) \text{ within } C$$

$$= 0$$

Hence from (1), we have $\frac{1}{2\pi i}\int_C \left[\frac{f'(z)+g'(z)}{f(z)+g(z)}\right] dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$

$\therefore$ $N_1 = N_2$ where N_1 and N_2 denote respectively the number of zeros of $f(z) + g(z)$ and $f(z)$ inside C. Hence the theorem.

Remark. We can deduce the Fundamental theorem of Algebra from Rouche's theorem.

Theorem 4: (Fundamental theorem of algebra)

A polynomial of degree n with complex coefficients has n zeros in $\boldsymbol{C}$.

Proof:

Let $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, be a polynomial of degree n.

Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 + \cdots + a_{n-1} z^{n-1}$.

Clearly $\lim\limits_{z\to\infty} \frac{g(z)}{f(z)} = 0$.

Hence there exists a positive real number r such that $\left|\frac{g(z)}{f(z)}\right| < 1$ for all z with

$|z| > r$.

Hence by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the circle $|z| = r + 1$. But 0 is a zero of multiplicity n for $f(z)$.

Hence the given polynomial $f(z) + g(z)$ also has n zeros.

Solved Problems

Problem 1. Evaluate $\int_C \frac{dz}{2z+3}$ where C is $|z| = 2$.

Solution:

$z = -\frac{3}{2}$ is the simple pole of $f(z)$ which lies inside the circle $|z| = 2$.

Res $\left\{f(z); -\frac{3}{2}\right\} = \lim\limits_{z\to -3/2} \frac{h(z)}{k'(z)}$ where $h(z) = 1$ and $k(z) = 2z + 3$.

$\because$ Res $\left\{f(z); -\frac{3}{2}\right\} = \frac{1}{2}$.

By residue theorem $\int_C f(z)dz = 2\pi i \left(\frac{1}{2}\right) = \pi i$.

yiter.

$$\int_C \frac{dz}{2z+3} = \int_C \frac{dz}{2\left(z + \frac{3}{2}\right)}$$

$$= \frac{1}{2}\int_C \frac{dz}{\left(z + \frac{3}{2}\right)}$$

$$= \frac{1}{2}(2\pi i)\left(\because \int_C \frac{dz}{z-a} = 2\pi i\right)$$

$$= \pi i$$

Problem 2: Evaluate $\int_C \frac{dz}{z^2 e^z}$ where $C = \{z : |z| = 1\}$.

Solution:

Given integral can be written as $\int_C f(z)dz$ where $f(z) = \frac{e^{-z}}{z^2}$.

$f(z)$ has pole of order 2 at $z = 0$ which lies inside the circle $|z| = 1$.

Let $g(z) = e^{-z}$. Hence $g'(z) = -e^{-z}$.

$\therefore$ By Lemma 4, Res $\{f(z); 0\} = \frac{g'(0)}{1!} = -1$.

By residue theorem $\int_C f(z)dz = 2\pi i(-1) = -2\pi i$.

Problem 3:

Evaluate $\int_C \frac{2+3\sin \pi z}{z(z-1)^2} dz$ where C is the square having vertices $3+3i, 3-3i, -3+3i, -3-3i$.

Solution:

Let $f(z) = \frac{2+3\sin \pi z}{z(z-1)^2}$. Here $z = 0$ is a simple pole and $z = 1$ is a double pole for $f(z)$ and both of them lie within C.

$$\text{Res}\{f(z); 0\} = \lim_{z\to 0} z\left(\frac{2+3\sin \pi z}{z(z-1)^2}\right) = 2$$

$$\text{Res}\{f(z); 1\} = \frac{g'(1)}{1!} \text{ where } g(z) = \frac{2+3\sin \pi z}{z}$$

$$g'(z) = \frac{z3\pi\cos \pi z - (2+3\sin \pi z)}{z^2}$$

$$\therefore\ g'(1) = -3\pi - 2.$$

$$\therefore\ \text{Res}\{f(z); 1\} = -3\pi - 2.$$

$\therefore$ By residue theorem $\int_C f(z)dz = 2\pi i(2 - 3\pi - 2) = -6\pi^2 i$.

Problem 4:

Evaluate $\int_C \tan z dz$ where C is $|z| = 2$.

Solution:

Let $f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{h(z)}{k(z)}$.

$\cos z$ has zeros at $z = \dfrac{(2n+1)\pi}{2}, n \in \mathbf{N}$.

$\therefore f(z)$ has simple poles at $z = -\frac{\pi}{2}$ and $z = \frac{\pi}{2}$ inside the circle $|z| = 2$.

$$\text{Res}\{f(z);\pi/2\} = \frac{h(\pi/2)}{k'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

$$\text{Res}\{f(z);-\pi/2\} = \frac{h(-\pi/2)}{k'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$

Problem 5:

Prove that $\int_C \frac{e^{2z}}{(z+1)^3}dz = \frac{4\pi i}{e^2}$ where C is $|z| = \frac{3}{2}$.

Solution:

Let $f(z) = \frac{e^{2z}}{(z+1)^3}$

$f(z)$ has a pole of order 3 at $z = -1$.

kes$\{f(z);-1\} = \frac{g''(-1)}{2!}$ where $g(z) = e^{2z}$.

Now $g'(z) = 2e^{2z}$ and $g''(z) = 4e^{2z}$.

Resi$\{f(z);-1\} = \frac{4e^{-2}}{2!} = \frac{2}{e^2}$.

By residue theorem $\int_C f(z)dz = 2\pi i\left(\frac{2}{e^2}\right) = \frac{4\pi i}{e^2}$.

Problem 6:

Evaluate, using (i) Cauchy's integral formula (ii) residue theorem $\frac{z+1}{z^2+2z+4}dz$ where C is the circle $|z+1+i| = 2$.

Solution:

Clearly C is a circle with centre $a = -(1+i)$ and radius 2.

Now
$$\frac{z+1}{z^2+2z+4} = \frac{z+1}{(z+1)^2+(\sqrt{3})^2}$$
$$= \frac{z+1}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}$$
$$= \frac{z+1}{[z-(-1-i\sqrt{3})][(z-(-1+i\sqrt{3})}$$

$z_0 = -1+i\sqrt{3}$ and $z_1 = -1-i\sqrt{3}$ are the singular points of the given integrand $\frac{z+1}{z^2+2z+4}$.

Now $|z_0 - a| = |i(\sqrt{3}+1)| = \sqrt{3}+1 > 2$

and $|z_1 - a| = |-i(\sqrt{3}-1)| = \sqrt{3}-1 < 2$.

$\therefore\ z_1 = -1 - i\sqrt{3}$ lies inside C.

(i) By using Cauchy integral formula.

Consider $f(z) = \frac{z+1}{z-(-1-i\sqrt{3})}$.

We note that $f(z)$ is analytic at all points inside C.

$\therefore$ By Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_1}dz = f(z_1)$;

(i.e.) $\frac{1}{2\pi i}\int_C \frac{(z+1)dz}{[z-(1-i\sqrt{3})][z-(-1+i\sqrt{3})]} = f(-1-i\sqrt{3})$

(i.e.) $\frac{1}{2\pi i}\int_C \frac{(z+1)dz}{z^2+2z+4} = \frac{-(-1-i\sqrt{3})+1}{(-1-i\sqrt{3})-(-1+i\sqrt{3})}$

$$\cdot = \frac{-i\sqrt{3}}{-2i\sqrt{3}}$$
$$= \frac{1}{2}$$

$$\therefore \int_C \frac{(z+1)dz}{z^2+2z+4} = \frac{1}{2}(2\pi i) = \pi i$$

(ii) By using residue theorem.

$$f(z) = \frac{z+1}{z^2+2z+4}$$

Since $z = -1 - i\sqrt{3}$ lies inside C

$\text{Res}\{f(z); -1-i\sqrt{3}\} = \frac{h(-1-i\sqrt{3})}{k'(-1-i\sqrt{3})}$ where $h(z) = z+1$ and

$k(z) = z^2 + 2z + 4$ so that $k'(z) = 2z + 2$

$$\therefore \text{Res}\{f(z); -1-i\sqrt{3}\} = \frac{-1-i\sqrt{3}+1}{2(-1-i\sqrt{3})+2} = \frac{-i\sqrt{3}}{-i2\sqrt{3}} = \frac{1}{2}$$

By residue theorem $\int_C f(z)dz = \frac{2\pi i}{2} = \pi i$.

Problem 7:

Use residue calculus to evaluate $\int_C \frac{3\cos z}{2i-3z} dz$ where C is the unit circle.

Solution:

Let $f(z) = \frac{3\cos z}{2i-3z}$.

Here $z = \frac{2i}{3}$ is a simple pole and lies within C.

$\text{Res}\left\{f(z); \frac{2i}{3}\right\} = \lim_{z \to 2i/3} \frac{h(z)}{k'(z)}$ where $h(z) = 3\cos z$ and $k(z) = 2i - 3z$ so that $k'(z) = -3$.

$$\therefore \ \text{Res}\{f(z); 2i/3\} = \frac{3\cos(2i/3)}{-3} = -\cos(2i/3) = -\cosh(2/3)$$

By residue theorem $\int_C f(z)dz = 2\pi i[-\cosh(2/3)]$.

(i.e.) $\int_C \frac{3\cos z}{2i-3z} dz = -2\pi i\cosh(2/3)$.

Problem 8:

Use residue theorem to evaluate $\int \frac{3z^2+z-1}{(z^2-1)(z-3)} dz$ around the circle

Solution:

Let $f(z) = \frac{3z^2+z-1}{(z^2-1)(z-3)}$.

$f(z)$ has simple poles $1, -1, 3$ and only $1, -1$ lie inside $|z| = 2$.

$\text{Res}\{f(z); 1\} = \frac{h(1)}{k'(1)}$ where $h(z) = 3z^2 + z - 1$ and $k(z) = z^3 - 3z^2 - z + 3$ so that $k'(z) = 3z^2 - 6z - 1$.

$\therefore \text{Res}\{f(z); 1\} = \frac{3+1-1}{3-6-1} = \frac{-3}{4}$.

Similarly, $\text{Res}\{f(z); -1\} = \frac{3-1-1}{3+6-1} = \frac{1}{8}$.

$\therefore$ By residue theorem,

$$\int \frac{3z^2 - z - 1}{(z^2 - 1)(z - 3)} dz = 2\pi i\left(-\frac{3}{4} + \frac{1}{8}\right) = 2\pi i\left(\frac{-5}{8}\right) = \frac{-5\pi i}{4}$$

Problem 9:

Evaluate $\int_C \frac{e^z dz}{(z+2)(z-1)}$ where C is the circle $|z-1| = 1$.

Solution:

$f(z) = \frac{e^z}{(z+2)(z-1)}$.

$f(z)$ has simple poles at $1, -2$; the pole 1 is inside the circle $|z-1| = 1$ and $z = -2$ lies outside the circle.

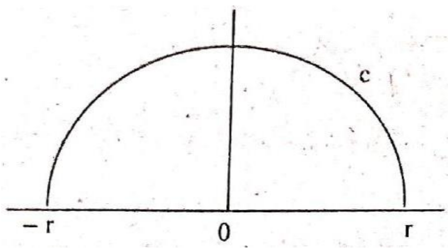
$$\text{Res}\{f(z); 1\} = \lim_{z\to 1}(z-1)\left(\frac{e^z}{(z+2)(z-1)}\right) = \frac{e}{3}.$$

By residue theorem $\int_C f(z)dz = 2\pi i\left(\frac{e}{3}\right)$.

$\therefore \int \frac{e^z}{(z+2)(z-1)} dz = \frac{i2\pi e}{3}$.

Problem 10:

Show that the function $2 + z^2 - e^{iz}$ has precisely one zero in the open upper half plane.



Solution:

Take $f(z) = 2 + z^2$ and $g(z) = -e^{iz}$. Let C be the simple closed curve consisting of the semi circle $|z| = r$ in the upper half plane together with the interval* $[-r, r]$ on the real axis.

If $z \in [-r, r]$ then $|g(z)| = 1$ and $|f(z)| \geq 2$.

Hence $|f(z)| > |g(z)|$.

Now, if $z = re^{i\theta}, \theta < \theta < \pi$ then $|f(z)| = |2 + z^2| \geq |z^2| - 2 = r^2 - 2$

Also $|g(z)| = \left|-e^{ire^{i\theta}}\right| = e^{-r\sin\theta}$.

Hence for sufficiently large value of r we have $|f(z)| > |g(z)|$.

Hence by Rouche's theorem $f(z) + g(z) = 2 + z^2 - e^{iz}$ and $f(z) = 2 + z^2$ have the same number of zeros in the upper half plane. Also $2 + z^2$ has exactly one zero in the upper half of the plane namely $i\sqrt{2}$.

Hence $2 + z^2 - e^{iz}$ has exactly one root in the upper half plane.

Problem 11:

Let $f(z) = \frac{(z^2+1)}{(z^2+2z+2)^2}$. Evaluate $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$ where C is the circle $|z| = 4$.

Solution:

i and $-i$ are zeros of order 1 and $-1 + i$ and $-1 - i$ are poles of order 2 for $f(z)$. Also, these zeros and poles lie inside C.

Hence number of zeros of $f(z) = N = 2$ and number of poles of $f(z) = P = 4$.

(Poles are counted according to their multiplicity)

$\therefore$ By Argument theorem $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P = 2 - 4 = -2$

Exercises:

1.Evaluate the following integrals.

(i) $\int_C \frac{3z-4}{z(z-1)}dz$ where C is the circle $|z| = 2$.

(ii) $\int_C \frac{(3z-4)}{z(z-1)(z-2)}dz$ where C is the circle $|z| = \frac{3}{2}$

(iii) $\int_C \frac{2z^2+4}{z^2-1}dz$ where (a) C is the circle $|z| = 2$ (b) C is the circle

$$|z - 1| = 1$$

(iv) $\int_C \frac{3dz}{z+1}$ where C is the circle $|z| = 2$:

(v) $\int_C \frac{3+z}{z}dz$ where C is the circle $|z| = 1$.

(vi) $\int_C \frac{z+1}{z^2-2z}dz$ where C is the circle $|z| = 3$.

(vii) $\int_C \frac{3z^3+2}{(z-1)(z^2+4)}dz$ where C is (a) $|z - 2| = 2$ (b) $|z| = 4$.

(viii) $\int_C \frac{dz}{z^3(z-1)}$ where C is the circle $|z| = 3$.

(ix) $\int_C \frac{e^{-z}}{z^2} dz$ where C is the circle $|z| = 1$.

(x) $\int_C \frac{dz}{z^3(z+4)} dz$ where C is (a) $|z| = 2$ (b) $|z + 2| = 3$.

2.Prove that $\int_C \coth z dz = 0$ where C is the circle $|z| = 1$.

3. Prove that $\int_C \coth z dz = 0$ where C is the circle $|z| = 1$.

4. Prove that $\int_C ze^{1/z} dz = \pi i$ where C is the circle $|z| = 5$.

5. Prove that $\int_C \frac{e^z dz}{\cosh z} = 8\pi i$ where C is the circle $|z| = 5$.

Answers.

1. (i) $6\pi i$ (ii) $-2\pi i$ (iii) (a) $2\pi i$ (b) $3\pi i$ (iv) $-2\pi i$

(v) $6\pi i$ (vi) $2\pi i$ (vii) (a) πi (b) $6\pi i$ (viii) $4\pi i$

(ix) $-2\pi i$ (x) (a) $\frac{\pi i}{32}$ (b) 0

5.3. Evaluation of Definite Integrals:

We use Cauchy's residue theorem for evaluating certain types of real definite integrals.

TYPE 1. $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. As θ varies from 0 to $2\pi, z$ describes the unit circle $|z| = 1$

Also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$ and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

Substituting these values in the given integrand the integral is transformed into $\int_C \theta(z) dz$ where $\theta(z) = f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$ and C is the positively oriented unit circle $|z| = 1$. The integral $\int_C \theta(z)dz$ can be evaluated using the residue theorem.

Solved problems

Problem 1:

Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Solution. Let $I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

Put $z = e^{i\theta}$.

Then $dz = izd\theta$ and $\sin\theta = \frac{z-z^{-1}}{2i}$.

The given integral is transformed to $I = \int_C \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]}$ where C is the unit circle $|z| = 1$. $= \int_C \frac{dz}{2z^2+5iz-2}$

Let $f(z) = \frac{1}{2z^2+5iz-2} = \frac{1}{2(z+2i)(z+i/2)}$

$\therefore$ $-2i$ and $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside C.

Also $\text{Res}\{f(z); -i/2\} = \lim_{z\to i/2} \frac{1}{2(z+2i)} = \frac{1}{3i}$.

Hence by Cauchy's residue theorem $I = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$.

Problem 2: Prove that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}, (-1 < a < 1)$

Solution:

Put $z = e^{i\theta}$. Then $\sin\theta = \frac{z-z^{-1}}{2i}$ and $dz = izd\theta$.

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_C \frac{dz}{iz\left[1+a\left(\frac{z-z^{-1}}{2i}\right)\right]}$$ where C is the unit circle

$$= \int_C \frac{2dz}{z[2i+a(z-z^{-1})]}$$

$$= \int_C \frac{2dz}{az^2+2iz-a}$$

Let $f(z)=\frac{2}{az^2+2iz-a}$The poles of $f(z)$ are given by $z=\frac{-2i\pm\sqrt{-4+4a^2}}{2a}$

$$=\frac{-i\pm i\sqrt{1-a^2}}{a}\text{(since }-1<a<1)$$

Let $z_1=\frac{-1+i\sqrt{1-a^2}}{a}$ and $z_2=\frac{-i-i\sqrt{1-a^2}}{a}$

We note that $|z_2|=\frac{1+\sqrt{1-a^2}}{|a|}>1$ (since $-1<a<1$)

Also, since $|z_1z_2|=1$ it follows the $|z_1|<1$. Hence there are no singular points on C and $z=z_1$ is the only simple pole insisde C

$$\operatorname{Res}\{f(z);z_1\}=\lim_{z\to z_1}(z-z_1)\left[\frac{2/a}{(z-z_1)(z-z_2)}\right]$$
$$=\frac{2/a}{z_1-z_2}$$
$$=\frac{1}{i\sqrt{1-a^2}}$$

By residue theorem $\int_0^{2\pi}\frac{d\theta}{1+a\sin\theta}=2\pi i\left[\frac{1}{i\sqrt{1-a^2}}\right]=\frac{2\pi}{\sqrt{1-a^2}}$

Problem 3: Prove that $I=\int_0^{\pi}\frac{ad\theta}{a^2+\sin^2\theta}=\frac{\pi}{\sqrt{a^2+1}}(a>0)$

Solution: $I=\int_0^{\pi}\frac{ad\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$

$$=\int_0^{\pi}\frac{2ad\theta}{2a^2+1-\cos 2\theta}$$
$$=\int_0^{2\pi}\frac{ad\varphi}{2a^2+1-\cos\varphi}$$
$$=\frac{1}{i}\int_C\frac{adz}{z\left[2a^2+1-\frac{(z+z^{-1})}{2}\right]}\text{ (putting }2\theta=\varphi)$$
$$=\frac{2a}{i}\int_C\frac{dz}{[2(2a^2+1)z-z^2-1]}$$
$$=2ai\int_C\frac{dz}{z^2-2(2a^2+1)z+1}$$
$$=2ai\int_C f(z)dz(1)$$

where $f(z) = \frac{1}{z^2-2(2a^2+1)z+1}$ and C is the unit circle $|z| = 1$.

Poles of $f(z)$ are the roots of $z^2 - 2(2a^2+1)z + 1 = 0$.

$\therefore\ z = (2a^2+1) \pm 2a\sqrt{a^2+1}$

Let $z_1 = (2a^2+1) + 2a\sqrt{a^2+1}$; $z_2 = (2a^2+1) - 2a\sqrt{a^2+1}$

Clearly $|z_1| > 1$ and $|z_1 z_2| = 1$ so that $|z_2| < 1$.

Hence the only pole inside C is $z = z_2$.

$$\text{Res}\,(f(z); z_2) = \lim_{z \to z_2} (z - z_2) \frac{1}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{1}{(-4a)\sqrt{a^2+1}}$$

From (1), $I = 2\pi i \left[-\frac{2ai}{-4a\sqrt{a^2+1}}\right]$

$$= \frac{\pi}{\sqrt{a^2+1}}$$

Problem 4:

Using Contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$.

Solution:

Let $I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$.

Put $z = e^{i\theta}$. Then $dz = ie^{i\theta} d\theta = izd\theta$.

Also $\sin\theta = \frac{z-z^{-1}}{2i}$. $\therefore$ The given integral is transformed to

$$I = \int_C \frac{dz}{iz\left[13 + 5\left(\frac{z-z^{-1}}{2i}\right)\right]}$$ (where C is the circle $|z| = 1$)

$$= \int_C \frac{dz}{iz\left[13 + 5\left(\frac{z^2-1}{i2z}\right)\right]}$$

$$= \int_C \frac{2dz}{5z^2 + 26iz - 5}$$

$-\frac{5}{5}$ and $-5i$ are simple poles of $f(z)$ and the pole $-\frac{i}{5}$ lies inside the unit circle.

$$\text{Res}\left\{f(z); -\frac{i}{5}\right\} = \lim_{z \to -i/5} \frac{h(z)}{k'(z)} \quad \begin{array}{l}(\text{ where } h(z) = 2 \text{ and} \\ k(z) = 5z^2 + i26z - 5)\end{array}$$

$$= \lim_{z \to -i/5} \left(\frac{2}{10z + i26}\right)$$

$$= \frac{2}{-2i + 26i}$$

$$= \frac{1}{12i}.$$

Hence by Cauchy's residue theorem $I = 2\pi i \left(\frac{1}{12i}\right) = \frac{\pi}{6}$.

Problem 5:

Use Contour integration technique to find the value of $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Put $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = izd\theta$.

Also $\cos\theta = \frac{z+z^{-1}}{2}$.

$\therefore$ The given integral is transformed to $I = \int_C \frac{dz}{iz\left[2+\frac{z+z^{-1}}{2}\right]}$ where C is the unit circle $|z| = 1$.

$$\therefore I = \int_C \frac{dz}{iz\left[2 + \left(\frac{z^2+1}{2z}\right)\right]}$$

$$= \int_C \frac{2dz}{i(4z + z^2 + 1)}$$

$$= \int_C \frac{-2idz}{z^2 + 4z + 1}$$

Let

$$f(z) = \frac{-2i}{z^2 + 4z + 1}$$

$$= \frac{-2i}{(z+2)^2 - 3}$$

$$= \frac{-2i}{(z + 2 - \sqrt{3})(z + 2 + \sqrt{3})}$$

$\therefore -2+\sqrt{3}$ and $-2-\sqrt{3}$ are simple poles of $f(z)$; the pole $-2+\sqrt{3}$ lies inside C.

$$\operatorname{Res}\{f(z); -2+\sqrt{3}\} = \lim_{z\to -2+\sqrt{3}} \left(\frac{-2i}{2z+4}\right)$$
$$= \frac{-2i}{-4+2\sqrt{3}+4}$$
$$= \frac{-i}{\sqrt{3}}$$

Hence by Cauchy's residue theorem $I = 2\pi i\left(\frac{-i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$.

Exercises

1.Show that $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$.

2.Show that $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta} = \frac{\pi}{6}$.

3.Show that $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$.

4.Show that $\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}$.

5.Show that $\int_0^{2\pi} \frac{d\theta}{\cos\theta+2\sin\theta+4} = \frac{2\pi}{\sqrt{11}}$.

6. Show that $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}} (a > 1)$.

7. Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} (a > b > 0)$.

8. Show that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}} (a^2 < 1)$.

9. Show that $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2a\pi}{(a^2-1)^{3/2}} (a > 1)$.

10. Show that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$.

11. Show that $\int_0^{2\pi} \frac{d\theta}{a^2-2a\cos\theta+1} = \frac{2\pi}{1-a^2} (0 < a < 1)$.

TYPE 2.

$\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{g(x)}{h(x)}$ and $g(x), h(x)$ are polynomials in x and the degree of $h(x)$ exceeds that of $g(x)$ by at least two.

To evaluate this type of integral we take $f(z) = \frac{g(z)}{h(z)}$.

The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0$.

Case (i) No pole of $f(z)$ lies on the real axis.

We choose the curve C consisting of the interval $[-r, r]$ on the real axis and the semi circle $|z| = r$ lying in the upper half of the plane.

Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C. Then we have

$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz$$

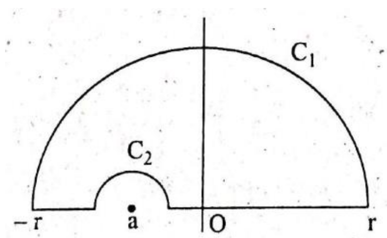
where C_1 is the semi circle.

Since $\deg h(x) - \deg g(x) \geq 2$ it follows thàt $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$ and hence $\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx$.

$\therefore \int_{-\infty}^{\infty} f(x)dx$ can be evaluated by evaluating $\int_C f(z)dz$ which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii) $f(z)$ has poles lying on the real axis.

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi-circle C_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small (refer figure).



Such an indenting must be done for every pole of $f(z)$ lying on the real axis.

It can be proved that $\int_{C_2} f(z)dz = -\pi i \text{Res}\{f(z); a\}$. By taking limit as $r \to \infty$ and $\varepsilon \to 0$ we obtain the value of $\int_{-\infty}^{\infty} f(x)dx$.

Solved Problems

Problem 1:

Use Contour integration method to evaluate $\int_0^\infty \frac{dx}{1+x^4}$.

Solution:

Let $f(z) = \frac{1}{1+z^4}$.

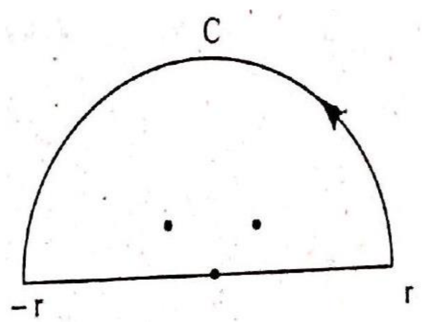
poles of $f(z)$ are given by the roots of the equation $z^4 + 1 = 0$, which are the four with roots of -1 .

De Moivre's theorem they are given by $e^{i\pi/4}; e^{i3\pi/4}; e^{i5\pi/4}; e^{i7\pi/4}$ and all are simple des.

de choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper emi-circle $|z| = r$ which we denote by C_1.

$$\therefore \int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \dots\dots(1)$$

The poles of $f(z)$ lying inside the contour C are obviously $e^{i\pi/4}$ and $e^{i3\pi/4}$ only. We find the residues of $f(z)$ at these points.



$\operatorname{Res}\{f(z); e^{i\pi/4}\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})}$ where $h(z) = 1$ and $k(z) = z^4 + 1$ so that $k'(z) = 4z^3$.

$$\therefore \operatorname{Res}\{f(z); e^{i\pi/4}\} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$$

Similarly, $\operatorname{Res}\{f(z); e^{i3\pi/4}\} = \frac{e^{-i9\pi/4}}{4}$.

By residue theorem

$$\int_C f(z)dz = 2\pi i(\text{ sum of the residues at the poles })$$

$$= 2\pi i\left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4}\right]$$

$$= \frac{\pi i}{2}[(\cos(3\pi/4) - i\sin(3\pi/4)) + (\cos(9\pi/4) - i\sin(9\pi/4))]$$

$$= \frac{\pi i}{2}\left[\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

From (1) $\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{C_1} f(z)dz = \frac{\pi}{2}$.

As $r \to \infty$, $\int_{C_1} f(z)dz \to 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$\therefore 2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}\left(\because \frac{1}{1+x^4} \text{ is an even function}\right).$$

$$\therefore \int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Problem 2:

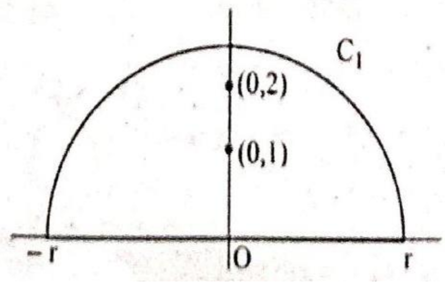
Using the method of contour integration evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$.

Solution:

Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$.

The poles of $f(z)$ are $i, -i, 2i, -2i$.

Choose the contour C as shown in the figure.



poles i and $2i$ lie within C. By residue theorem

$$\int_C f(z)dz = 2\pi i(\text{ sum of the residues of } f(z)) \ldots \ldots \ldots (1)$$

find the residues of $f(z)$.

$\{f(z); i\} = \frac{h(i)}{k'(i)}$ where $h(z) = z^2$ and $k(z) = (z^2+1)(z^2+4) = z^4 + 5z^2 + 4$ that $k'(z) = 4z^3 + 10z$.

$$\therefore \text{Res}\{f(z); i\} = \frac{-1}{-4i + 10i} = \frac{-1}{6i} = \frac{i}{6}$$

$$\text{Res}\{f(z); 2i\} = -\frac{i}{3} \text{ (verify)}$$

$$\therefore \text{ From (1) } \int_C f(z)dz \quad = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right)$$

$$= -2\pi i \cdot \left(\frac{-i}{6}\right)$$

$$= \frac{\pi}{3} \ldots\ldots\ldots\ldots(2)$$

$\therefore$ Also (1) can be written, using (2), as

$$\int_{C_1} f(z)dz + \int_{-r}^{r} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3} \ldots \ldots \ldots \ldots (3)$$

Further the integral $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$$

Problem 3:

Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{a+b}$.

Solution:

Proceed as in previous problem.

Problem 4:

Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx$.

Solution:

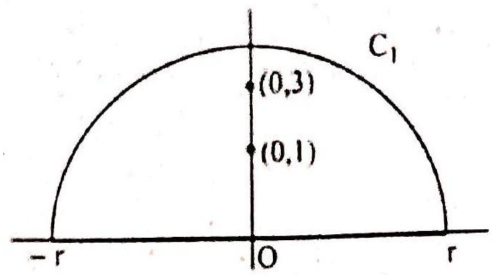
Let $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$.

Pcles of $f(z)$ are the zeros of $z^4 + 10z^2 + 9 = 0$.

$z^4 + 10z^2 + 9 = (z^2 + 9)(z^2 + 1)$

$\therefore z = \pm 3i; \pm i$. Hence $z = 3i, -3i, i - i$ are the simple poles of $f(z)$.

Choose the contour C as shown in the figure.



$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \quad \dots\dots\dots (1)$$

The poles of $f(z)$ lying within C are i and $3i$ and both of them are simple poles.

$\text{Res}\{f(z); i\} = \frac{h(i)}{k'(i)}$ where $h(z) = z^2 - z + 2$ and

$$k(z) = z^4 + 10z^2 + 9 \text{ so that } k'(z) = 4z^3 + 20z$$

$$\therefore \text{Res}\{f(z); i\} = \frac{-1 - i + 2}{-4i + 20i} = \frac{1 - i}{16i}.$$

Similarly, Res $\{f(z); 3i\} = \frac{7+3i}{48i}$ (verify).

$$\therefore \int_C dz = 2\pi i(\text{ sum of the residues at the poles })$$

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

(1) $\int_{-r}^{r} \frac{x^2-x+2}{x^4+10x^2+9} dx + \int_{C_1} \frac{z^2-z+2}{z^4+10z^2+9} dz = \frac{5\pi}{12}$.

$a^s l \to \infty$ the integral over $C_1 \to 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx = \frac{5\pi}{12}$$

Problem 5:

Evaluate $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$.

Solution:

Since $\frac{1}{(x^2+a^2)^2}$ is an even function, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = 2\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$$

$$f(z) = \frac{1}{(z^2+a^2)^2}$$

poles of $f(z)$ are the roots of $(z^2+a^2)^2 = 0$.

Now, $(z^2+a^2)^2 = (z+ai)^2(z-ai)^2$.

ai and - ai are double poles of $f(z)$.

hoose the contour C consisting of the interval $[-r, r]$ on the real axis and the semi icle C_1 with centre 0 and radius r that lies in the upper half plane.

$$\therefore \int_{-r}^{r} f(x)dx + \int_{C_l} f(z)dz = \int_C f(z)dz \ldots\ldots\ldots.(1)$$

The poles of $f(z)$ lying within C is $z = ai$.

$\text{Res}\{f(z); ai\} = \frac{1}{1!} g'(ai)$ where $g(z) = \frac{1}{(z+ai)^2}$.

low $g'(z) = -2(z+ai)^{-3}$.

$g'(ai) = \frac{1}{4a^3 i}$.

$\therefore \text{Res}(f(z); a_i) = \frac{1}{4a^3 i}$.

$\therefore \int_C f(z)dz = 2\pi i \left(\frac{i}{4a^3 i}\right) = \frac{\pi}{2a^3}$.

$\therefore \int_{-r}^{r} \frac{dx}{(x^2+a^2)^2} + \int_{C_1} f(z)dz = \frac{\pi}{2a^3}$.

When $r \to \infty$ the integral over $C_1 \to 0$.

$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$.

$\therefore \int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$.

Problem 6:

Prove that $\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$.

Solution:

Since $\frac{1}{x^6+1}$ is an even function we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^\infty \frac{dx}{x^6+1}$$

Now, let $f(z) = \frac{1}{z^6+1}$.

The poles of $f(z)$ are given by the roots of the equation $z^6 + 1 = 0$, which are the sixth roots of -1 .

By De Moivre's theorem they are given by $e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e^{i7\pi/6}$ and $e^{i11\pi/6}$ and they are simple poles.

Now choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi circle $|z| = r$, which we denote by C_1.

The poles of $f(z)$ lying inside C are $e^{i\pi/6}, e^{i3\pi/6}$ and $e^{i5\pi/6}$.

$$\text{Res}\{f(z); e^{i\pi/6}\} = \frac{h(e^{i\pi/6})}{k'(e^{i\pi/6})} \quad \text{where } h(z) = 1 \text{ and } k(z) = z^6 + 1 \text{ so that } k'(z) = 6z^5$$

$$= \frac{1}{6e^{i5\pi/6}} = \frac{1}{6} e^{-i5\pi/6}$$

Similarly,Res$\{f(z); e^{i3\pi/6}\} = \frac{1}{6}e^{-15\pi/2}$ and

Res$\{f(z); e^{i5\pi/6}\} = \frac{1}{6}e^{-i25\pi/6}$.

By residue theorem

$$\int_C f(z)dz = 2\pi i \text{ (sum of the residues at the 3 poles)}$$

$$= 2\pi i\left[\frac{1}{6}e^{5i\pi/6} + \frac{1}{6}e^{-5i\pi/2} + \frac{1}{6}e^{-25i\pi/6}\right]$$

$$= \frac{2\pi i}{6}\left[\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}\right)\right.$$
$$\left. + \left(\cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6}\right)\right]$$

$$= \frac{\pi i}{3}\left[\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) + (0 - i) + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)\right]$$

$$= \frac{\pi i}{3}(-i - i) = \frac{2\pi}{3}$$

$\therefore$ From (1), $\int_{-r}^{r}\frac{dx}{x^6+1} + \int_{C_1} f(z)dz = \frac{2\pi}{3}$.

As $r \to \infty$ the integral over $C_1 \to 0$.

$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3}$.

$\therefore \int_{0}^{\infty}\frac{dx}{x^6+1} = \frac{\pi}{3}$.

Problem 7:

Prove that $\int_0^{\infty}\frac{x^4dx}{x^6-1} = \frac{\pi\sqrt{3}}{6}$.

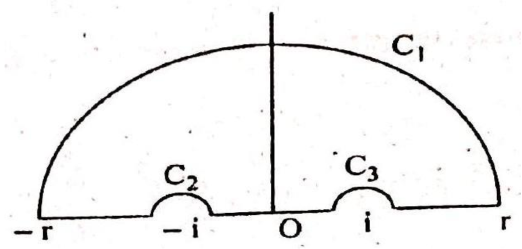
Solution:

Let $f(z) = \frac{z^4}{z^6-1}$.

The poles of $f(z)$ are given by the sixth roots of unity, namely $e^{2n\pi i/6}; n = 0,1, \ldots ,5$.

$\therefore f(z)$ has 2 simple poles on the real axis, viz., 1 and -1 and the two poles $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie on the upper half of the plane.

Now choose the contour C as shown in the figure.



$$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{-r}^{-1-\epsilon_1} f(x)dx + \int_{C_2} f(z)dz$$
$$+ \int_{-1+\epsilon_1}^{1-\epsilon_2} f(x)dx + \int_{C_3} f(z)dz + \int_{1+\epsilon_2}^{r} f(x)dx \ldots \ldots (1)$$

Now, $\int_{C_2} f(z)dz = -\pi i \text{Res}\{f(z); -1\}$

$= -\pi i \left(\frac{h(-1)}{k'(-1)}\right)$ where $h(z) = z^4$

$= -\pi i(-1/6)$

$= \frac{\pi i}{6} \quad \ldots \ldots \ldots \ldots (2)$

Similarly

$$\int_{C_3} f(z)dz = -\pi i \text{Res}\{f(z); 1\}$$

$= -\pi i \left(\frac{h(1)}{k'(1)}\right)$

$= -\pi i(1/6)$

$= -\frac{\pi i}{6} \ldots\ldots\ldots (3)$

$$\int_C f(z)dz = 2\pi i\left[\operatorname{Res}\{f(z); e^{\pi i/3}\} + \operatorname{Res}\{f(z); e^{2\pi i/3}\}\right]$$

$$= 2\pi i\left[\frac{h\left(e^{i\pi/3}\right)}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

$$= 2\pi i\left[\frac{e^{i4\pi/3}}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

Also

$$= \frac{\pi i}{3}\left(e^{-i\pi/3} + e^{-i2\pi/3}\right)$$

$$= \frac{\pi i}{3}\left(e^{-i\pi/3} - e^{i\pi/3}\right)$$

$$= \frac{\pi i}{3}(-2i\sin\pi/3)$$

$$= \frac{\pi\sqrt{3}}{3} \text{.........(4)}$$

substituting (2), (3), (4) in (1) and taking limits as $\epsilon_1, \epsilon_2 \to 0$ and $r \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{x^4}{x^6 - 1}dx + \frac{\pi i}{6} - \frac{\pi i}{6} = \frac{\pi\sqrt{3}}{3}$$

$$\therefore 2\int_0^{\infty} \frac{x^4}{x^6 - 1}dx = \frac{\pi\sqrt{3}}{3}$$

$$\therefore \int_0^{\infty} \frac{x^4}{x^6 - 1}dx = \frac{\pi\sqrt{3}}{6}$$

Exercises

1.Prove the following by using Cauchy's residue theorem.

(i) $\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$ (ii) $\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$

(iii) $\int_0^{\infty} \frac{x^2dx}{(x^2+1)^2} = \frac{\pi}{4}$ (iv) $\int_0^{\infty} \frac{dx}{x^4+x^2+1} = \frac{\pi\sqrt{3}}{6}$

(v) $\int_0^{\infty} \frac{(2x^2-1)dx}{x^4+5x^2+4} = \frac{\pi}{4}$ (vi) $\int_0^{\infty} \frac{dx}{(x^2+1)^3} = \frac{\pi}{16}$

(vii) $\int_0^{\infty} \frac{x^2dx}{(x^2+a^2)^3} = \frac{\pi}{16a^3}$ (viii) $\int_0^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$

(ix) $\int_0^{\infty} \frac{x^2dx}{(x^2+4)(x^2+9)} = \frac{\pi}{200}$ (x) $\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

2.Show that $\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+2)} = \frac{-\sqrt{2}\pi}{6}$.

TYPE 3.

$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax dx$ or $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax dx$ where $g(x)$ and $h(x)$ are real polynomials such that degree of $h(x)$ exceeds that of $g(x)$ by at least one and $a > 0$.

Case (i) $h(x)$ has no zeros on the real axis.

In this case take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$.

$\therefore f(z)$ has no poles on the real axis.

Choose the contour as in Type 2 and proceeding as in Type 2 we get the value of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iux} dx$.

Taking the real and imaginary parts of $\frac{g(x)}{h(x)} e^{iax} dx$ we obtain the values of

$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax dx$ and $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax dx$

Case (ii) $h(x)$ has zeros of order one on the real axis.

Take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$. We notice that $f(z)$ has real poles. Suppose a is a real zero of $h(x)$ on the real axis. In this case we indent the real axis as Case (ii) of Type 2 and evaluate the integral.

To prove that the integral over the upper semicircle tends to zero as $r \to \infty$, we use the following lemma.

for 4 Lemma. Let $f(z)$ be a function of the complex variable z satisfying the nuidg conditions.

(i) $f(z)$ is analytic in upper half plane except at a finite number of poles.

(i) $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \le \arg z \le \pi$.

(i) a is a positive integer.

Then $\lim_{r\to\infty} \int f(z) e^{iaz} dz = 0$ where C is the semi circle with centre at the origin and radius r.

Problem 1:

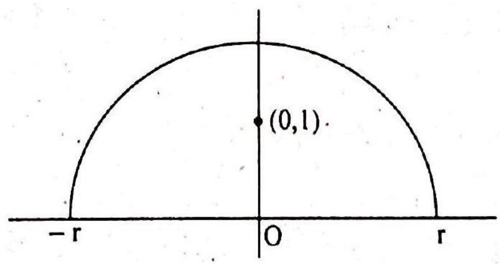
Prove that $\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$.

Solution:

Let $f(z) = \frac{e^{iz\gamma}}{1+z^2}$.

The poles of $f(z)$ are given by i and $-i$.

Choose the contour C as shown in the figure.



The pole of $f(z)$ that lies within C is i. Hence by residue theorem

$$\int_C f(z)dz = 2\pi i \text{Res}\{f(z); i\}$$

$$= 2\pi i \frac{h(i)}{k'(i)} \text{ where } h(z) = e^{iz} \text{ and } k(z) = 1 + z^2$$

$$= \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}$$

$\therefore \int_{-r}^{r} \frac{e^{iax}}{x^2+1}dx + \int_{C_1} \frac{e^{iaz}}{z^2+1}dz = \frac{\pi}{c}$.

When $r \to \infty$ the integral over C_1 tends to zero.

$\therefore \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx = \frac{\pi}{e}$.

Equating real parts we get $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}$.

$$\therefore 2\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}\left(\text{ since } \frac{\cos x}{1+x^2} \text{ is an even function }\right)$$

$$\therefore \int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$$

Problem 2:

Using the method of contour integration evaluate

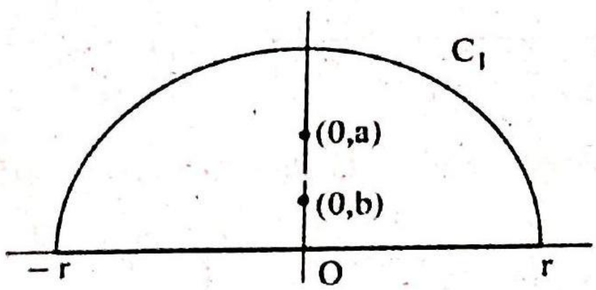
$$\int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx(a > b > 0)$$

Solution:

Let $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$.

The poles of $f(z)$ are $ia, -ia, ib, -ib$.

Choose the contour C as shown in the figure.



The poles of $f(z)$ which lie within C are ia and ib.

Hence by residue theorem

$$\int_C f(z)dz = 2\pi i\left(\text{ sum of the residues of } f(z)\right) \ldots\ldots\ldots.(1)$$

We find the residues of $f(z)$.

$\text{Res}(f(z); a_l) = \frac{h(ai)}{k'(ai)}$ where $h(z) = e^{iz}$ and

b $= (z^2+a^2)(z^2+b^2) = z^4 + (a^2+b^2)z^2 + a^2b^2$ so that

$(b) = 4z^3 + 2(a^2+b^2)z$.

$$\therefore \text{Res}[f(z); ai] = \frac{e^{-a}}{4(ia)^3 + 2(a^2 + b^2)(ia)}$$
$$= \frac{e^{-a}}{i2a[(a^2 + b^2) - 2a^2]}$$
$$= \frac{-ie^{-a}}{2a(b^2 - a^2)}$$
$$= \frac{-ie^{-a}}{2a(a^2 - b^2)}$$

similarly, $\text{Res}\{f(z); bi\} = \frac{ie^{-b}}{2b(b^2-a^2)}$

$$= \frac{-ie^{-b}}{2b(a^2 - b^2)}.$$

From (1) $\int_C f(z)dz = 2\pi i \left[\frac{i}{2(a^2 - b^2)}\left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b}\right)\right]$

$$= \frac{-\pi}{a^2 - b^2}\left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b}\right) = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \ldots..(2)$$

Also (1) can be written using (2) as

$$\int_{C_1} f(z)dz + \int_{-r}^{r} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Further the integral over C_1 tends to 0 as $r \to \infty$.

$\therefore$ (3) becomes

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating real parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Problem 3:

Prove that $\int_0^\infty \frac{\cos ax dx}{(x^2+1)^2} = \frac{\pi}{4}(a + 1)e^{-a}$ where $a > 0$.

Solution:

Let $f(z) = \frac{e^{iaz}}{(z^2+1)^2}$.

The poles of $f(z)$ are given by i and $-i$ which are double poles. Now choose the conto as in Problem 1. The pole of $f(z)$ that lies within C is i.

Now, $\operatorname{Res}\{f(z); i\} = \frac{1}{1!} g'(i)$ where $g(z) = (z-i)^2 f(z) = \frac{e^{iaz}}{(z+i)^2}$.

$$\therefore g'(z) = \frac{(z+i)^2 iae^{iaz} - e^{iaz} 2(z+i)}{(z+i)^4}$$

$$\therefore \operatorname{Res}\{f(z); i\} = \frac{-4iae^{-a} - e^{-a}(4i)}{16} = \frac{-ie^{-a}(a+1)}{4}$$

Hence by Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \left(\frac{ie^{-a}(a+1)}{4}\right) = \frac{\pi(a+1)e^{-a}}{2}$$

$$\therefore \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz = \frac{\pi(a+1)e^{-a}}{2}$$

As $r \to \infty$, the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \frac{\pi(a+1)e^{-a}}{2}$$

Equating real parts $\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)^2} dx = \frac{\pi(a+1)e^{-u}}{2}$.

$$\therefore \int_0^{\infty} \frac{\cos ax}{(x^2+1)^2} dx = \frac{\pi(a+1)e^{-a}}{4}$$

Problem 4:

Prove that $\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2+4x+5} = -\frac{\pi \sin 2}{e}$.

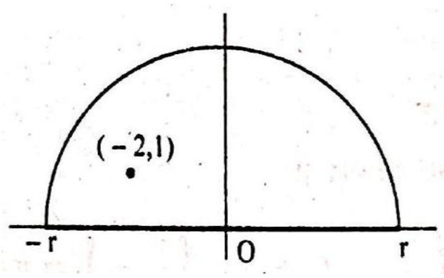
Solution:

Let $f(z) = \frac{e^{iz}}{z^2+4z+5}$.

he poles of $f(z)$ are the roots of the equath $z^2 + 4z + 5 = 0$. They are given by

$$z = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i.$$

low, choose the contour $\boldsymbol{C}$ as shown in the figure.



$-2+i$ is the only pole of $f(z)$ that lies within C and it is a simple pole.

Hence by Cauchy' residue theorem,

$$\int_C f(z)dz = 2\pi i \mathrm{Res}\{f(z); -2+i\}$$

$$= (2\pi i)\frac{h(-2+i)}{k'(-2+i)} \quad \text{where } h(z) = e^{iz}$$

$$\int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz = \frac{\pi e^{-2i}}{e}$$

Since the integral over C_1 tends to zero as $r \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi e^{-2i}}{e} = \frac{\pi}{e}(\cos 2 - i\sin 2)$$

Equating imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2+4x+5} = -\frac{\pi \sin 2}{e}$$

Problem 5:

Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2} dx$.

Solution:

Let $f(z) = \frac{ze^{iz}}{z^2+a^2}$

The poles of $f(z)$ are given by ia and $-ia$ which are simple poles. Choose the conto C as in Problem 1.

Only the pole $z = ia$ lies inside C.

$\text{Res}\{f(z); ia\} = \frac{h(ia)}{k'(ia)}$ where $h(z) = ze^{iz}$ and $k(z) = z^2 + a^2$ so that $k'(z) = 2z$.

$\therefore \text{Res}\{f(z); ia\} = \frac{iae^{i(ia)}}{2(ia)} = \frac{e^{-a}}{2}$.

Hence by Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i\left(\frac{e^{-a}}{2}\right) = \pi i e^{-a}$$

$$\therefore \int_{-r}^{r} f(x)dx + \int_{C_l} f(z)dz = \pi i e^{-a}$$

When $r \to \infty, \int_{C_1} f(z)dz = 0$.

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \pi i e^{-a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx = \pi i e^{-a}$$

$$\text{(i.e.)} \int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)dx}{x^2 + a^2} = \pi i e^{-a}$$

$$\text{(i.e.)} \int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)dx}{x^2 + a^2} = \pi i e^{-a}$$

Equating imaginary parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 + a^2} dx = \pi e^{-a}$$

Hence the above integrand is an even function we have

$$2\int_{0}^{\infty} \frac{x\sin x}{x^2 + a^2} dx = \pi e^{-a}$$

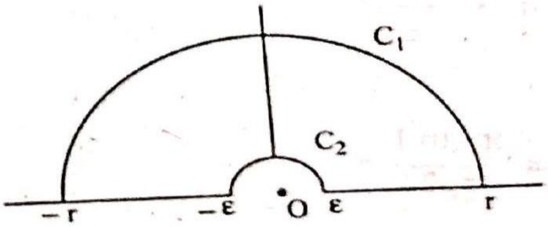
$$\therefore \int_{0}^{\infty} \frac{x\sin x}{x^2 + a^2} dx = \frac{\pi e^{-a}}{2}$$

Problem 6: Prove that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Solution:

Let $f(z) = \frac{e^{iz}}{z}$.

The only singular point of $f(z)$ is 0 which is a simple pole and it lies on the real axis. Now choose the contour C as shown in the figure.



Then $\int_C f(z)dz = \int_{-r}^{-\epsilon} f(x)dx + \int_{C_2} f(z)dz$

$$+\int_{\epsilon}^{r} f(x)dx + \int_{C_1} f(z)dz \dots\dots\dots (1)$$

Since $f(z)$ is analytic within C, $\int_C f(z)dz = 0 \dots\dots (2)$

$$\text{Also } \int_{C_2} f(z)dz = -\pi i \text{Res}(f(z):0)$$

$$= -\pi i e^0$$

$$= -\pi i \dots\dots\dots\dots (3)$$

Further the integral over C_1 tends to 0 as $r \to \infty$.

Hence using (2) and (3) in (1) and taking limit as $r \to \infty$ we get

$0 = \int_{-\infty}^{0} f(x)dx - \pi i + \int_0^{\infty} f(x)dx.$

$\therefore \int_{-\infty}^{\infty} f(x)dx = \pi i.$

Equating the imaginary parts we get $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$

$\therefore \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ (since $\frac{\sin x}{x}$ is an even function).

Exercises

1.Establish the following with the help of residues.

(i) $\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2+4x+5} = -\frac{\pi \sin 2}{e}$

(ii) $\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2-2x+5} = \frac{\pi \sin 1}{2e^2}$

(iii) $\int_0^\infty \frac{\cos ax dx}{(x^2+b^2)^2} = \frac{\pi}{4b^3}(1 + ab)e^{-ab}(a > 0, b > 0)$.

(iv) $\int_0^\infty \frac{x\sin x dx}{x^2+a^2} = \frac{\pi e^{-a}}{2}(a > 0)$.

(v)$\frac{\cos x dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)(a > b > 0)$.

(vi)$\frac{8}{(a\cos x+x\sin x)dx} = 2\pi e^{-a}$.

(vii)$\int \frac{x\sin ax}{x^4+4}dx = \frac{\pi e^{-a}\sin a}{2}$.

Wind the following with the help of residues.

(i)$\int_0^\infty \frac{\sin mx dx}{x} = \frac{\pi}{2}(m > 0)$ (ii) $\int \frac{\cos mx}{x}dx = 0(m > 0)$

(iii) $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$ (iv) $\int_{-\infty}^\infty \frac{\cos mx}{x-a}dx = -\pi\sin(ma)$.

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